Nilpotence in Finitary Linear Groups

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1. Results and Examples

Throughout this paper F denotes a field and V a vector space over F. The *finitary general linear group* $\mathrm{FGL}(V) = \mathrm{F\,Aut}_F V$ over V is the subgroup of $\mathrm{Aut}_F V$ of F-automorphisms g of V such that $[V, g] = V(g-1)$ is finite-dimensional over F. A *finitary linear group* is a subgroup of $\mathrm{FGL}(V)$ for some F and V. We can always choose F as large as we please, algebraically closed for example; for if E is an extension field of F then there is an obvious embedding of $\mathrm{F\,Aut}_F V$ into $\mathrm{F\,Aut}_E(E \otimes_F V)$.

There has been much interest of late in the group-theoretic structure of finitary linear groups, with works of Hall [4] and Meierfrankenfeld, Phillips, and Puglisi [5] especially of note. In particular, [5] analyses the solubility structure of such groups. Here we carry out an analogous exercise for nilpotence, except that we are interested not just in nilpotent and locally nilpotent groups, but also in the various canonical locally nilpotent and hypercentral normal subgroups and the four canonical Engel sets of an *arbitrary* finitary linear group.

The following theorem summarizes our main positive conclusions. Our notation, which we explain in detail immediately after the statement of the theorem, is standard (with the exception of the introduction of $\eta_2(G)$), following [9] or [10] for example.

THEOREM. *Let F be a field of characteristic $p \geq 0$, V a vector space over F, and G any subgroup of* $\mathrm{FGL}(V)$.

(a) $L(G) = \eta(G) = \sigma(G) = \eta_2(G) = \langle M \lhd G : M = \zeta_{\omega 2}(M) \rangle = \langle M \lhd G : \exists \alpha < \omega 2,\ M = \zeta_\alpha(M) \rangle$.

(b) $\bar{L}(G) = \bar{\sigma}(G) = \eta_1 \eta(G) = \eta_1(G)$.

(c) $R(G) = \rho(G) \geq \zeta(G)$. *For each finite subset X of G there is a normal subgroup K of G with $K \supseteq X$ and $R(G) \cap K \leq \zeta_{\omega 2}(K)$.*

(d) $\bar{R}(G) = \bar{\rho}(G) \geq \zeta_\omega(G)$. *For each finite subset X of G there is a normal subgroup K of G with $K \supseteq X$ and $\bar{R}(G) \cap K \leq \zeta_\omega(K)$.*

(e) *Modulo its unipotent radical, G has central height at most $\omega 2$.*

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