Cut Loci, Minimizers, and Wavefronts in Riemannian Manifolds with Boundary

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This note investigates, in the setting of Riemannian manifolds with boundary, the ideas that in ordinary Riemannian manifolds fall under the heading of "cut locus". The new features, which we illustrate by examples, are due to the fact that minimizers can bifurcate and merge. Thus there may exist open sets each of whose points p has the property that every point q sufficiently far from p has more than one minimizer from p. We show (Theorem 1) that for any p and almost all q, there is a natural way to choose exactly one minimizer from p to q. This simple construction applies uniformly to all complete, connected Riemannian manifolds with boundary. In consequence, we may extend a substantial part of the classical theory to manifolds with boundary.

THEOREM 1. *Let p be a point of a complete, connected Riemannian manifold with boundary M. The set of points that have two minimizers from p with distinct terminal velocity vectors has measure zero in M. The complement of this set can be expressed as a union of "primary" minimizers displaying tree-like branching behavior.*

Throughout, M will denote a connected, metrically complete, and C^∞ Riemannian manifold with C^∞ boundary. *Geodesics* will be locally minimizing curves parameterized proportionally to arclength by $[0,1]$. Recall that geodesics are C^1 and that any two points of M are joined by a minimizer. For a general reference on geodesics in Riemannian manifolds with boundary, see [ABB1]. Cut loci in manifolds with boundary were previously investigated by Wolter in his dissertation [W2], to which we refer below.

1. Primary Minimizers

It is shown in [ABB2] that geodesics can bifurcate only by *involution,* that is, by unrolling from the boundary. The considerable difficulty of the proof is due to the fact that a geodesic may contact the boundary in, say, a Cantor-like set. We now state this theorem precisely; with a view to its use here, it is stated as a uniqueness theorem for given terminal position and velocity.

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