Convexity and the Schwarz–Christoffel Mapping

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1. Introduction

In 1952 the author wrote an article [2] on close-to-convex functions. The present paper shows how additional conclusions about univalent functions can be obtained from the results and methods in [2]. We also provide a new proof of the main result of [2] with the aid of the support angle function of Study.

For a discussion of close-to-convex functions one is referred to [1], especially pages 46–51. We recall that a function $f(z)$ analytic in the unit disc Δ is called *close-to-convex* if $\mathrm{Re}(f'(z)/\phi'(z))>0$ for some convex function $\phi(z)$ in Δ. Every close-to-convex function is necessarily univalent.

In [2], the following theorem is proved:

THEOREM A. *Let $f(z)$ be locally univalent in Δ. Then f is close-to-convex in Δ if and only if*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} d\theta > -\pi, \quad z=re^{i\theta}, \tag{1}$$

for each r, $0<r<1$, and each pair of real numbers θ_1, θ_2 with $\theta_1<\theta_2$.

From Theorem 3 of [2], one deduces the following theorem.

THEOREM B. *If $f(z)\neq$ constant is analytic in Δ and continuous on $\bar{\Delta}$ and $u=\mathrm{Re}[f(z)]$ is monotone nondecreasing as z moves around $\partial\Delta$ from z_0 to $z_1\neq z_0$ in the positive direction and monotone nonincreasing as z moves around $\partial\Delta$ from z_1 to z_0 in the positive direction, then f is close-to-convex in Δ.*

2. The Support Angle Function

A basic tool will be the support angle function (Stützwinkelfunktion) introduced by Study [4, p. 89]. If f is analytic in Δ and locally univalent, then for the disc $|z|\le\rho<1$ $(0<\rho<1)$, a support angle function is

$$S_{f,\rho}(\theta)=p_f(\rho,\theta)+\theta, \quad -\infty<\theta<\infty, \tag{2}$$

Received May 30, 1991.
Michigan Math. J. 40 (1993).