# On Vinogradov's Mean Value Theorem, II

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## 1. Introduction

The main purpose of this note is to provide an improvement of Vinogradov's mean value theorem which may be of use in multiplicative number theory. Let $J_{s,k}(P)$ denote the number of solutions of the simultaneous diophantine equations

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = 0 \quad (1 \le j \le k) \tag{1}$$

with $1 \le x_i, y_i \le P$ for $1 \le i \le s$. On writing

$$f(\underline{\alpha}; Q) = \sum_{x \le Q} e(\alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_k x^k), \tag{2}$$

in which $e(\alpha)$ denotes $e^{2\pi i \alpha}$, we observe that

$$J_{s,k}(P) = \int_{\mathbf{T}^k} |f(\underline{\alpha}; P)|^{2s} \, d\underline{\alpha}, \tag{3}$$

where $\mathbf{T}^k$ denotes the $k$-dimensional unit cube and $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$. Estimates for the mean value (3) were first investigated by Vinogradov, and are now known collectively as *Vinogradov's mean value theorem*. These estimates have found varied uses in both additive and multiplicative number theory.

Modern bounds for $J_{s,k}(P)$ take the form

$$J_{rk,k}(P) \le D(k,r) P^{2rk - \frac{1}{2}k(k+1) + \eta(r,k)} \quad (r \in \mathbf{N}), \tag{4}$$

where $D(k,r)$ is independent of $P$, and

$$\eta(r,k) = \tfrac{1}{2}k^2(1 - 1/k)^r. \tag{5}$$

The most general bound currently in the literature appears to be due to Stechkin [5], who showed that when $k \ge 2$ the bound (4) holds with (5) for each $P \in \mathbf{R}^+$ and $r \in \mathbf{N}$, with

$$D(k,r) = \exp(C \min\{r,k\} k^2 \log k) \tag{6}$$

and $C$ an absolute constant. The explicit nature of the constant (6) is of importance when it comes to obtaining zero-free regions for the Riemann zeta

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