Surfaces Parameterizing Waring Presentations of Smooth Plane Cubics

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1. Introduction

Let V be an n-dimensional vector space over an algebraically closed field k of characteristic 0, and let ϕ be a homogeneous polynomial (form) of degree d on V. A Waring presentation is a presentation of ϕ as a sum of m dth powers of linear forms on V. In this paper we shall study the variety of all such presentations in the case $d=3$, $n=3$, and $m=4$. To introduce the questions we will be discussing, we first consider the case of quadratic forms where the answers are easy and well known.

A nonsingular quadratic form Q on V can be written as a sum of n squares of linear forms $l_i \in V^*$:

$$Q = l_1^2 + \cdots + l_n^2.$$

The variety parameterizing all such n-tuples $(l_1, \ldots, l_n)$ for a given quadratic form Q is isomorphic to the orthogonal group $O_n(k)$. Thus this variety has two isomorphic irreducible components, corresponding to orthogonal matrices of determinant $+1$ and -1 respectively. It follows from the classical Cayley formulas that these components are rational. Indeed, let $so_n(k)$ be the set of all skew-symmetric complex $n \times n$ matrices and let I be the $n \times n$ identity matrix. Then the following mutually inverse rational maps

$$\begin{array}{ccc} so_n(k) & \rightleftharpoons & SO_n(k) \\ A & \to & (I+A)(I-A)^{-1} \\ (B-I)(I+B)^{-1} & \leftarrow & B \end{array}$$

establish a birational isomorphism between $SO_n(k)$ and the affine space $so_n(k)$.

Suppose we start with a cubic form ϕ on a 3-dimensional vector space V which cuts out a smooth curve C in $P(V)$. Assume that the j-invariant of C is nonzero, that is, ϕ cannot be written as a sum of three cubes of linear forms. Then ϕ can be written as a sum of four cubes:

$$\phi = l_1^3 + \cdots + l_4^3; \tag{1}$$

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