Factorization of Blaschke Products

KEIJI IZUCHI

1. Introduction

Let H^∞ be the space of bounded analytic functions in the open unit disc D. Identifying these functions with boundary functions, we can consider that H^∞ is the essentially supremum norm closed subalgebra of L^∞, the space of bounded measurable functions on ∂D with respect to the Lebesgue measure. Sarason [14] proved that $H^\infty + C$ is a closed subalgebra of L^∞, where C is the set of continuous functions on ∂D. We denote by $M(H^\infty + C)$ the maximal ideal space of $H^\infty + C$. In [6], Guillory and Sarason proved that there is a positive integer N such that if $f \in H^\infty + C$ and b is an inner function with $|f| \le |b|$ on $M(H^\infty + C)$, then $f^N/b = f^N \bar{b}$ belongs to $H^\infty + C$, and we cannot take $N = 1$. In [12], the author and Y. Izuchi proved that we can take $N = 2$. In this paper, we assume that b is a Blaschke product and study the cases $|f| \le |b|$ on $M(H^\infty + C)$ and $f\bar{b} \notin H^\infty + C$. Our aim is to investigate the kind of small changes of f or b, say g and ψ respectively, that make $g\bar{b} \in H^\infty + C$ or $f\bar{\psi} \in H^\infty + C$. To prove several of our theorems, Hoffman's factorization theorem for Blaschke products [9, Thm. 5.2] plays an important role.

In Section 3, we shall give an additional property in Hoffman's factorization theorem that zero sets of its factors having zeros of infinite order coincide with each other. In Section 4, we prove that if $f \in H^\infty + C$ and b is a Blaschke product with $|f| \le |b|$ on $M(H^\infty + C)$, then there is a subproduct ψ of b such that $f\bar{\psi} \in H^\infty + C$ and $Z(\psi) = Z(b)$, and there is a function g in $H^\infty + C$ such that $|g| = |f|$ on $M(H^\infty + C)$ and $g\bar{b} \in H^\infty + C$. In Section 5, we shall give a sufficient condition for which the absolute moduli of two Blaschke products coincide on $M(H^\infty + C)$.

2. Preliminaries

For a sequence $\{z_n\}_n$ of points in D with $\sum_{n=1}^\infty 1 - |z_n| < \infty$, the function

$$b(z) = \prod_{n=1}^\infty \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D,$$

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