Neighborhoods of S^1-like Continua in 4-Manifolds

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0. Introduction

In this paper we study the problem of determining which compact subsets of 4-manifolds have close neighborhoods that collapse to 1-dimensional spines. As is explained in [10], the study of this problem is motivated by the desire to understand engulfing of 2-dimensional polyhedra in piecewise linear 4-manifolds. The technology of 4-manifold topology does not seem to be well enough developed for us to characterize such compacta completely. We restrict our attention, therefore, to the case in which the neighborhood collapses to a copy of the circle, S^1. In that case the fundamental groups which arise are infinite cyclic, so that we can apply the $\mathbf{Z}$-theory of Freedman and Quinn [2; 3]. Our main theorem characterizes those compact subsets of 4-manifolds that have arbitrarily close neighborhoods with spines homeomorphic to S^1.

THEOREM 1. *Suppose X is a compact subset of the orientable 4-manifold M^4. Then X has arbitrarily close neighborhoods homeomorphic to $S^1 \times B^3$ if and only if*

(1) *X has the shape of some S^1-like continuum, and*
(2) *X satisfies the inessential loops condition.*

Let Y be an S^1-like continuum. Then Y is the inverse limit of an inverse sequence in which each space is S^1. Thus there is a standard embedding of Y in S^4 as the intersection of a nested sequence of thin tubes, each tube homeomorphic with $S^1 \times B^3$. We will identify Y with this embedded copy of Y. The following complement theorem is then a corollary to Theorem 1. We use $\mathrm{Fd}(X)$ to denote the fundamental dimension of X.

COROLLARY. *Suppose X is a compact subset of S^4, $\mathrm{Fd}(X)=1$, X satisfies the inessential loops condition, and Y is an S^1-like continuum standardly embedded in S^4. Then $S^4 - X \cong S^4 - Y$ if and only if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.*

Received October 26, 1988. Revision received July 6, 1992.
[1] Research partially supported by University of Alabama Faculty Research Grant number 1453.
[2] Research partially supported by National Science Foundation Grants DMS-8701791 and DMS-8900822.
Michigan Math. J. 40 (1993).