A Conjecture of L. Carleson and Applications

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1. Introduction

Let m be the area measure on $\mathbf{C}$. For a meromorphic function f in the unit disc U, let

$$A(r,f)=\int_{\{|z|\le r\}}\frac{|f'|^2}{(1+|f|^2)^2}\,dm,\quad 0\le r<1, \tag{1.1}$$

be the spherical area of the image of $\{|z|\le r\}$ by f, counting multiplicities. In his thesis Carleson [6] considered the classes T_α, $0\le\alpha<1$, of meromorphic functions f in U satisfying

$$|f|_\alpha=\int_0^1 A(r,f)(1-r)^{-\alpha}\,dr<\infty, \tag{1.2}$$

and the class T_1 of meromorphic functions f in U with the property that $A(r,f)$ remains bounded when r tends to 1, that is,

$$|f|_1=\sup_{r<1}A(r,f)<\infty. \tag{1.3}$$

We obviously have $T_1\subset T_\alpha\subset T_\beta\subset T_0$ for all $\alpha,\beta\in(0,1)$ with $\alpha>\beta$. The class T_0 coincides with the class of functions with bounded characteristic, and a well-known theorem of F. and R. Nevanlinna asserts that each $f\in T_0$ is the quotient of two bounded analytic functions in U. In [6, p. 39] Carleson proved an analogue of this theorem for the classes T_α just defined, namely, the fact that each function in T_α is the quotient of two bounded functions, each of which is in T_β for all $\beta<\alpha$, and conjectured that one cannot take $\beta=\alpha$, that is, not every function in T_α is the quotient of two bounded functions in T_α. For all $\alpha\in[0,1]$, T_α contains the weighted Dirichlet space $D_{1-\alpha}$ of analytic functions f in U satisfying

$$\int_U|f'(z)|^2(1-|z|)^{1-\alpha}\,dm<\infty \tag{1.4}$$

Recently, in their paper [11] on invariant subspaces of the multiplication operator on the Dirichlet space D_0, Richter and Shields found a partial "negative" answer to Carleson's conjecture for $\alpha=1$ by showing that every function in

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