Length Functions and Outer Space

JOHN SMILLIE* & KAREN VOGTMANN**

1. Introduction

Let F_n be the free group of rank n. The outer automorphism group of F_n acts on a certain contractible space called *Outer Space* which was introduced in [CV1]. The role played by Outer Space in the study of the group of outer automorphisms of F_n is analogous to the role played by Teichmüller space in the study of the mapping class group of a surface. In [T] Thurston constructs an embedding of Teichmüller space into a *finite-dimensional* projective space by means of length functions. In this paper we show that a similar construction for Outer Space is not possible.

We say a graph has genus n if its fundamental group is isomorphic to F_n. An $\mathbf{R}$-graph is a graph which is a metric space where each edge is isometric to an interval of $\mathbf{R}$. Outer Space can be thought of as a normalized collection of marked $\mathbf{R}$-graphs of genus n with no free edges, where a *marking* on a graph is an identification of the fundamental group of the graph with F_n, and where two graphs are equivalent if there is an isometry between them which preserves the marking. There are two methods of implementing the normalization. Define the total length of a graph to be the sum of the lengths of the edges. The first method of normalization is to consider only graphs of total length 1. An alternative method of normalization is to consider equivalence classes of graphs where two graphs are equivalent if there is a homeomorphism preserving the marking taking one to the other which multiplies lengths by a constant.

The term *length function* is used in two different ways in combinatorial group theory. There are the length functions introduced by Lyndon (in [CM] these are called "based length functions") and there are the "hyperbolic length functions" of [AB] which are called "translation length functions" in [CM]. In our context an element of Outer Space determines a hyperbolic length function on F_n, where the length of $g \in F_n$ is the length of the shortest (unbased) loop representing the homotopy class corresponding to g. (The definition of the Lyndon length function involves choosing a base point and

Received March 1, 1991.
*Partially supported by NSF grant #DMS-9003101.
**Partially supported by NSF grant #DMS-8702070.
Michigan Math. J. 39 (1992).