Localization of Hilbert Modules

XIAOMAN CHEN & R. G. DOUGLAS

In [6] it was shown how to reformulate the study of bounded linear operators on Hilbert spaces into the study of Hilbert modules. Here we study modules over a function algebra which have also a Hilbert space structure relative to which the multiplication is jointly continuous.

Localization is a useful method in the study of Hilbert modules. Many important invariants can be obtained via localization. For example, one can recover the characteristic operator function for the canonical model of Sz.-Nagy and Foiaş as well as the curvature of some hermitian holomorphic vector bundles. Some examples [6] show that localization seems to be related to the index and spectral theory of Hilbert modules.

Localization of Hilbert modules was introduced in [6] using the module tensor product. Tensoring with a finite-dimensional module whose spectrum consists of a single point yields another finite-dimensional module with the same spectrum. Analyzing the latter module yields the sought-after invariants.

A finite-dimensional module with spectrum a single point involves various partial derivatives and their evaluation at that point. One defines the order of such a module to be the order of the highest derivative that is needed.

In this note, we show that zero$^{\text{th}}$ order localization is not enough to get much information about a Hilbert module. In particular, in the several-variables case (which is our main interest in the Hilbert module approach to operator theory), zero$^{\text{th}}$ order localization determines just the first stage of the Koszul complex introduced by Taylor [7]. In [5], first-order localization was considered and some interesting calculations were made. However, much groundwork needs to be developed to obtain efficient localization techniques for finding invariants for Hilbert modules via localization.

In this paper, we confine our attention to Hilbert modules over $A(\Omega)$, $A(\Omega)=\{f \mid f\in C(\bar{\Omega})\cap \mathrm{Hol}(\Omega)\}$, where Ω is a bounded connected domain in $\mathbf{C}^n$. We define higher-order localizations for this kind of Hilbert module. As an application we study the relation between modules being locally unitarily equivalent and globally unitarily equivalent. We obtain an extension of the Cowen–Douglas theory [1; 2] to the context of Hilbert modules.

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