On the Lebesgue Test for the Convergence of Vilenkin–Fourier Series

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1. Introduction

In the 1970's Onneweer and Waterman obtained analogues, in the context of bounded Vilenkin groups, for the Salem test [1] and the Lebesgue test [2] for the convergence of Fourier series. Recently, by localizing the Salem test, Waterman found a new criterion for the pointwise convergence of Fourier series [3]. Here we adapt Waterman's localization of the Salem test to obtain an extension of the Onneweer–Waterman Lebesgue test for convergence of Fourier series of functions defined on a bounded Vilenkin group.

2. Notation and Terminology

By a *Vilenkin group* G we mean a compact, 0-dimensional, metrizable abelian group. G contains a fundamental system $\{G_k\}$ of neighborhoods of 0 such that

(i) $G=G_0\supset G_1\supset\cdots\supset G_k\supset G_{k+1}\supset\cdots\supset\{0\}$;
(ii) the quotient G_{k-1}/G_k is of prime order p_k;
(iii) $\{0\}=\bigcap_{k=0}^{\infty} G_k$.

If the sequence of primes $\{p_k\}$ is a bounded sequence, we say that G is a *bounded Vilenkin group*; otherwise we say that G is *unbounded.* If $p_k=2$ for every k, then we will call G the *Walsh group* and denote G by 2^{ω}.

Set $m_0=1$ and $m_i=\prod_{\nu=1}^{i} p_\nu$. Each $n\in\mathbf{Z}^+$ has a unique representation $n=\sum_{i=0}^{s} a_i m_i$ with $0\le a_i<p_{i+1}$ for $0\le i\le s$. Denoting the dual of G by X, we may enumerate the elements χ_n of X in such a way that

$$\chi_n=\chi_{m_0}^{a_0}\cdot\chi_{m_1}^{a_1}\cdot\dots\cdot\chi_{m_s}^{a_s}.$$

For each $k\in\mathbf{Z}^+$ there is an $x_k\in G_k\backslash G_{k+1}$ such that $\chi_{m_k}(x_k)=e^{2\pi i/p_{k+1}}=\zeta_k$. Each $x\in G$ has a unique representation $x=\sum_{i=0}^{\infty} b_i x_i$ with $0\le b_i<p_{i+1}$. We can enumerate the cosets of G_k in G by means of the lexicographic ordering of the coset representatives of the form $z=\sum_{i=0}^{k-1} b_i x_i$ $(0\le b_i<p_{i+1})$. In particular, we shall let $z_\alpha^{(k)}$ be that coset representative z for which $\alpha=\sum_{i=0}^{k-1} b_i(m_k/m_{i+1})$.

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