Rational Powers of Generators of Möbius Groups

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1. Introduction

In this paper we will be concerned with 2-generator Fuchsian groups, in particular groups of the first kind (see [4] or [1]; also, for 2-generator groups see e.g. [10] or [12]). We will use the following notation: The group of all Möbius self-maps of the upper half plane $\mathbf{H}=\{z\in\mathbf{C}: \operatorname{Im}(z)>0\}$ will be denoted by $\mathrm{PSL}(2,\mathbf{R})$, and Möbius self-maps of the Riemann sphere will be denoted by $\mathrm{PSL}(2,\mathbf{C})$.

We need a definition of roots and rational powers of Möbius maps; this is motivated by the fact that, for each integer n, the nth power of g in $\mathrm{PSL}(2,\mathbf{R})$, written g^n, is its nth iterate. Thus we want an nth root of g to be a map in $\mathrm{PSL}(2,\mathbf{R})$ whose nth iterate is g; this will be denoted $g^{1/n}$ and will not in general be unique. We similarly define rational powers of g. We begin with the simplest case, namely rational powers of parabolic Möbius maps. Rational powers of hyperbolic maps are also easy to define but will not be needed here.

DEFINITION. Let g_0 be the parabolic Möbius map given by $g_0: z\mapsto z+1$, and let g in $\mathrm{PSL}(2,\mathbf{R})$ also be parabolic. Thus there is a map f in $\mathrm{PSL}(2,\mathbf{C})$ with $g=fg_0f^{-1}$. For each k in $\mathbf{Q}$ we define the kth *power* of g_0 to be $g_0^k: z\mapsto z+k$ and the kth power of g to be $g^k=fg_0^kf^{-1}$.

We remark that it is necessary for f to be in $\mathrm{PSL}(2,\mathbf{C})$ and not just in $\mathrm{PSL}(2,\mathbf{R})$. The above definition gives a unique kth power of a parabolic Möbius map for all rational k. Hyperbolic maps in $\mathrm{PSL}(2,\mathbf{R})$ also have unique rational powers. Rational powers of elliptic maps are not unique, and are defined as follows.

DEFINITION. Let g_0 be the elliptic Möbius self-map of the unit disc $g_0: z\mapsto e^{i\theta}z/e^{-i\theta}$ where $0<|\theta|\le\pi/2$, and let g be an elliptic element of $\mathrm{PSL}(2,\mathbf{R})$ whose trace satisfies $\mathrm{tr}^2(g)=\mathrm{tr}^2(g_0)=4\cos^2\theta$. Thus there is an f in $\mathrm{PSL}(2,\mathbf{C})$ with $g=fg_0f^{-1}$. For each rational k we define a kth power of g_0 to be $g_0^k: z\mapsto e^{ki\theta}z/e^{-ki\theta}$ and a kth power of g to be $g^k=fg_0^kf^{-1}$.

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