Besov Spaces, Sobolev Spaces, and Cauchy Integrals

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1. Introduction and Statement of Results

Let B_n denote the unit ball in C^n with boundary S, the $(2n-1)$-dimensional sphere. If $d\sigma$ is normalized rotation invariant measure on S and $f \in L^1(d\sigma)$ then for $z \in B_n$ we define the Cauchy integral

$$Cf(z) = \int_S f(\zeta) \frac{d\sigma(\zeta)}{(1-\langle z, \zeta \rangle)^n}.$$

In this paper we obtain conditions on f sufficient to imply that Cf belongs to either the Besov space B_β^p or the Hardy–Sobolev space H_β^p, where $\beta > 0$ and $1 < p < \infty$. Recall that a holomorphic function F defined on B_n belongs to H_β^p if

$$\|F\|_{H_\beta^p}^p = \|R^\beta F\|_p^p = \sup_{0<r<1} \int_S |R^\beta F(r\zeta)|^p \, d\sigma(\zeta) < \infty,$$

where R^β denotes the radial fractional derivative operator defined on the class of harmonic functions on B_n by

$$R^\beta u(z) = \sum (1+k)^\beta P_k(z),$$

where $u = \sum P_k(z)$ is the expansion of u in homogeneous harmonic polynomials. Thus, if $z = r\zeta$ where $0 \le r < 1$ and $\zeta \in S$,

$$\begin{aligned} R^1 u(z) &= u(z) + r \frac{\partial u}{\partial r}(r\zeta) \\ &= u(z) + \sum_{j=1}^n \left(z_j \frac{\partial u}{\partial z_j} + \bar{z}_j \frac{\partial u}{\partial \bar{z}_j} \right). \end{aligned}$$

A holomorphic function F defined on B_n belongs to B_β^p if

$$\|F\|_{B_\beta^p}^p = \|R^{1+\beta} F\|_{p,p-1}^p = \int_{B_n} |R^{1+\beta} F(z)|^p (1-|z|)^{p-1} \, d\nu(z) < \infty,$$

where $d\nu$ denotes $2n$-dimensional Lebesgue measure defined on C^n.

The sufficient conditions we establish are of two types. The first we describe as "transverse" and the second we call "tangential". The transverse

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