Isomorphic Operator Algebras and Conjugate Inner Functions

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I. Introduction

Let D denote the open unit disk in the complex plane, $D=\{z: |z|<1\}$, and let m be normalized arclength measure on the boundary ∂D of D. If ϕ is a nonconstant inner function on D, then $C=C_\phi$ denotes the composition operator on $H^2=H^2(D)$ determined by ϕ—$C_\phi(f)=f\circ\phi$. Here $\circ$ denotes function composition. That C_ϕ is bounded is proven in [7; 8]. The operator C_ϕ does not tell everything about the analytic function ϕ. Indeed, if e_n is the function $e_n(z)=z^n$, then $C_{e_n}(e_m)=e_{nm}$ so that, for $n>1$, C_{e_n} is the direct sum of a 1-dimensional identity operator and a pure isometry of infinite multiplicity. As such, they are all unitarily equivalent to each other. On the other hand, e_n covers the disk n times so that these functions are not the same.

Each f in H^∞ defines the analytic Toeplitz operator T_f on H^2 by $T_f(h)=fh$. Let $\mathbf{A}=\mathbf{A}_\phi$ denote the norm closed algebra generated by C_ϕ and all the analytic Toeplitz operators. Note that $C_\phi T_f=T_{f\circ\phi}C_\phi$, so that $\mathbf{A}$ is commutative just in case ϕ is the identity function $\phi(z)=z$. From here on, the same notation will be used to denote the H^∞ function, its boundary function, its Toeplitz operator, and even its Gelfand transform. This convention is convenient and will cause no confusion.

Two inner functions ϕ and ψ are conjugate if there is an analytic homeomorphism τ of D satisfying $\tau\circ\psi=\phi\circ\tau$. We prove the following:

THEOREM 1. *If ϕ and ψ are nonconstant, nonperiodic inner functions, then they are conjugate if and only if the algebras $\mathbf{A}_\phi$ and $\mathbf{A}_\psi$ are isomorphic.*

Here, ϕ is periodic if $\phi^{(n)}(z)=z$, where $\phi^{(n)}$ denotes the n-fold iterate of ϕ. The analytic homeomorphisms of D are the Möbius transformations

$$\tau(z)=c\frac{z-a}{1-\bar{a}z},$$

where $|a|<1$ and $|c|=1$. Theorem 1 is just the analytic version of what is done in [1; 2; 4; 5] for composition operators on L^2 spaces.

If τ is a homeomorphism as in the theorem, then $C_\tau C_\phi C_\tau^{-1}=C_\psi$ and $C_\tau f C_\tau^{-1}=f\circ\tau$, so that the map $\Gamma(a)=C_\tau a C_\tau^{-1}$ is an isomorphism of $\mathbf{A}_\phi$

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