The Inclusion of Classical Families in the Closure of the Universal Teichmüller Space

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Introduction

A major problem of Teichmüller theory is to characterize the closure of Bers's model of the universal Teichmüller space $\mathfrak{I}$. In this paper, we give new geometric conditions that if satisfied by the range of a conformal mapping from a disk guarantee that the Schwarzian derivative of that mapping is in the closure of $\mathfrak{I}$. The condition is described by defining a subclass of close-to-convex functions. This result extends the work of Gehring and Astala [3].

Let $\mathbf{B}$ denote the Banach space of functions ϕ holomorphic on the unit disk $\mathbb{D}$ with finite norm

$$\|\phi\| = \sup_{z \in \mathbb{D}} |\phi(z)|[1-|z|^2].$$

If f is conformal on $\mathbb{D}$ and

$$\phi = \frac{d}{dz} \log f'(z) = \frac{f''(z)}{f'(z)},$$

then $\|\phi\| \leq 8$ (see [13]). The development of injectivity criteria for functions on $\mathbb{D}$ led Becker and Pommerenke (see [4; 5]) to study the space

$$\mathbf{T} = \{\phi = (f''/f') : f \text{ conformal on the unit disk } \mathbb{D} \text{ with quasiconformal extension to the Riemann sphere } \hat{\mathbb{C}}\}.$$

This subset of $\mathbf{B}$ is an alternative model for the universal Teichmüller space (see [3]).

In [3], Astala and Gehring develop a characterization of the closure of $\mathbf{T}$ in the $\|\cdot\|$ norm. They apply this characterization to prove that $f''/f' \in \mathrm{cl}(\mathbf{T})$ if f is convex in the direction θ. We introduce a new method which shows that $f''/f' \in \mathrm{cl}(\mathbf{T})$ provided that $\mathbb{C} \setminus f(\mathbb{D})$ is the union of disjoint half-lines l, where the angle of l is continuous in the prime end topology.

Let $\mathbf{L}$ denote the class of linearly accessible functions; that is, $\mathbf{L} = \{f$ conformal on $\mathbb{D}$ with $f(0) = 0$, $f'(0) = 1$: $\mathbb{C} \setminus f(\mathbb{D})$ is the union of closed half-lines such that the corresponding open half-lines are disjoint$\}$. Lewandowski [10; 11] (also see [6]) shows that $\mathbf{L}$ is equivalent to the class of close-to-convex functions introduced by Kaplan [9]. Thus, given $f \in \mathbf{L}$, there exists

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