Erdös–Turán Inequalities for Distance Functions on Spheres

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Dedicated to Professor B. Volkmann on the occasion of his 60th birthday

0. Introduction

When studying how much the distribution of an N-point subset ω_N of the unit interval $[0,1)$—or, equivalently, the unit circle in the complex plane—deviates from uniform distribution, we are led to a natural measure of deviation called the *discrepancy* $D(\omega_N)$ of ω_N. A classical result of Erdös and Turán relates this number to the maximum modulus $M(\omega_N)$ of the corresponding polynomial $p(z,\omega_N)$ on the unit circle, that is, the monic polynomial whose zeros are the points of ω_N.

Roughly speaking, the Erdös–Turán inequality states that a "small" value of $M(\omega_N)$ implies a certain degree of uniformity of the point set ω_N, expressed by a "small" value of $D(\omega_N)$. In the present paper we prove similar inequalities for a large class of distance functions defined for finite subsets ω_N of the unit sphere S^{d-1} in d-dimensional Euclidean space ($d \geq 2$). In essence, these inequalities relate the spherical cap discrepancy to certain "potentials" and also to certain "energy sums" generated by the set ω_N.

In Section 1 we prove two refinements of the classical Erdös–Turán inequality. First, by studying the L^1-norm of the function $\log|p(z,\omega_N)|$ instead of its maximum, an inequality is obtained which is best possible in some sense. Secondly, by considering the discrepancy *function* of the given set, we are in a position to account properly for irregularities of distribution that are "global" rather than "local".

In Section 2 we discuss the asymptotics of $\pi(\omega_N)$, the product of mutual distances between points of sets ω_N which are obtained by letting $\omega_N := \{z_1, \dots, z_N\}$, that is, the set of the first N terms of a fixed infinite sequence on the unit circle. We prove that, among all sequences, the van der Corput sequence essentially shows the best behaviour.

In Section 3 the classical Erdös–Turán inequality is generalized to an arbitrary dimension $d \geq 2$: We replace one-dimensional discrepancy by spherical

Received May 9, 1989. Revision received January 23, 1991.

† On March 10, 1990, Gerold Wagner lost his life in an avalanche in the Tyrolian Alps. Thereafter, the present paper was revised by Alexander Durner, Mathematisches Institut A, Universität Stuttgart, Postfach 80 11 40, D-7000 Stuttgart 80.

Michigan Math. J. 39 (1992).