Lewy Unsolvability and Several Complex Variables

LEON EHRENPREIS

I. Introduction

The history of surjectivity of linear partial differential operators L may be thought of as beginning with the Cauchy-Kowalewski theorem, which shows that if L has analytic coefficients then L maps analytic functions locally onto analytic functions. Thus, in the local analytic realm, the equation

(1-1) $$Lf = g$$

always has a solution.

Many "special" surjectivity $\mathcal{C}^\infty$ theorems were proved (e.g., for elliptic or hyperbolic L), but it took the development of the theory of distributions by L. Schwartz to yield a general surjectivity theorem independent of type. It was proven by Malgrange [15] and Ehrenpreis [4] in 1953 that surjectivity in (1-1) holds in the space of $\mathcal{C}^\infty$ functions, locally or globally (at least on convex sets), for operators with constant coefficients (see also [2]).

It was very surprising, therefore, that surjectivity on the $\mathcal{C}^\infty$ level (even locally) fails for an operator L that is first order and has linear coefficients. This was discovered by Lewy [13] in 1956. Lewy's operator is

(1-2) $$L = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - 2i(x+iy)\frac{\partial}{\partial t}$$

in the three variables x, y, t.

Somewhat later, Mizohata gave the example in two variables:

(1-3) $$M = \frac{\partial}{\partial x} + ix\frac{\partial}{\partial t}.$$

Starting with Lewy's original paper, many proofs have been given for the unsolvability of (1-1). Some of these proofs, such as Hörmander's [10], have led to vast generalizations.

In this paper we shall present two new proofs of the unsolvability of L. We shall see that each proof puts L in a new setting and leads to an interesting theory.

Received December 14, 1988. Revision received May 15, 1990.
Work partially supported by NSF grant number MCS 80-04486.
Michigan Math. J. 38 (1991).