Totally Real Submanifolds in a 6-Sphere

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1. Introduction

On a 6-dimensional unit sphere S^6, one can construct an almost complex structure using the properties of the Cayley division algebra; we refer to [3] for this construction. Further, it is known that this almost complex structure on S^6 is not integrable and that it is a nearly Kaehler structure on S^6. Regarding submanifolds of S^6, it is known that S^6 has no 4-dimensional complex submanifolds [4]. However, S^6 has 3-dimensional totally real submanifolds, which are minimal and orientable [3]. For a compact 3-dimensional totally real submanifold M of S^6, in [2] it is shown that if the sectional curvatures k of M satisfy $1/16 < k \leq 1$, then $k = 1$, that is, M is totally geodesic. However, there are 3-dimensional compact totally real submanifolds of S^6 some of whose sectional curvatures are greater than 1 (cf. [1, p. 436]).

The object of the present paper is to prove the following.

THEOREM. *Let M be a compact 3-dimensional totally real submanifold of S^6. If k_0 is the infimum of the sectional curvatures of M, then either $4k_0 \leq 1$ or M is totally geodesic.*

2. Totally Real Submanifolds of S^6

Let J be the almost complex structure defined on S^6 by the properties of the Cayley division algebra, and let g be the standard metric of constant curvature 1 on S^6. Then we have

$$g(JX, JY) = g(X, Y), \quad (\bar{\nabla}_X J)(X) = 0, \quad X, Y \in \chi(S^6), \tag{2.1}$$

where $\bar{\nabla}$ is the Riemannian connection on S^6 with respect to g and $\chi(S^6)$ is the Lie algebra of vector fields on S^6.

Define a tensor field G of the type $(1,2)$ on S^6 by $G(X, Y) = (\bar{\nabla}_X J)(Y)$, $X, Y \in \chi(S^6)$. This tensor field has the following properties:

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