A Density Criterion for Frames of Complex Exponentials

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1. Introduction

The notion of a frame has been introduced by Duffin and Schaeffer in [1]. It can be defined in a general Hilbert space H as follows. A sequence (e_n) of vectors of H is a *frame* if there exist positive constants C_1 and C_2 such that, for all f in H,

$$C_1\|f\|^2 \leq \sum |\langle f \mid e_n\rangle|^2 \leq C_2\|f\|^2. \tag{1}$$

Frames are important in the study of complex exponentials (cf. [1] and the book of R. M. Young on nonharmonic Fourier series [3]).

The following problem will be studied in this paper. Let $\Lambda = (\lambda_n)_{n\in Z}$ be a sequence of distinct real numbers. What is the upper bound of all numbers R such that the sequence of functions $(e^{i\lambda_n t})$ is a frame of $L^2([-R, R])$? This number, denoted $R(\Lambda)$, will be called the *frame radius* of the sequence Λ. Partial results were found by Duffin and Schaeffer [1] and Landau [2]. They are summarized in Theorems 1 and 2. The goal of the present paper is to give a necessary and sufficient condition for Λ to have a strictly positive finite frame radius, and, when it does, to obtain a formula for that radius.

We shall consider only sequences with distinct λ_n's since the general case can be dealt with as follows. The frame radius of the sequence λ_n is not changed if we repeat some λ_n's a finite and uniformly bounded number of times. If the number of repetitions is not bounded, the functions $(e^{i\lambda_n t})$ can never be a frame on any interval. Note also that, if the sequence of functions $(e^{i\lambda_n t})$ is a frame for the interval I, it is also a frame for each subinterval of I.

The reference space is $L^2(I)$, where I is a finite interval, and the inner product is given by

$$\langle f \mid g\rangle = \frac{1}{|I|}\int_I f(t)\bar{g}(t)\,dt,$$

where $|I|$ denotes the length of the interval. We denote by C, C_1, and C_2 constants which can change from one line to the next.

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