On Consecutive kth Power Residues, II

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1. Introduction

Brauer [1] proved that for any positive integers k and l and every sufficiently large prime p there exists a positive integer r such that the numbers $r, r+1, \dots, r+l-1$ are all kth power residues modulo p. Let $r(k,l,p)$ be the least such integer and define

$$\Lambda(k,l) = \limsup_{p\to\infty} r(k,l,p).$$

The function $\Lambda(k,l)$ has been studied by a number of authors. For example, it is known ([4], [8]) that $\Lambda(k,l)=\infty$ for $l\geq 4$ and all $k\geq 2$ and for $l=3$ and all even values of k. On the other hand, using machine computation it was shown that $\Lambda(k,2)$ is finite for every $k\leq 7$, and it has been conjectured [2] that the same is true for $k>7$ (see [7] for further references). In [7] we proved this conjecture for the case when k is a prime number. Here we shall prove the conjecture in full.

THEOREM 1. $\Lambda(k,2)<\infty$ *for all positive integers* k.

Stated differently, the assertion of the theorem is that, given a positive integer k, there exists a constant $c_0(k)$ such that for every sufficiently large prime p there exists a pair $(r, r+1)$ of consecutive kth power residues modulo p satisfying $1\leq r\leq c_0(k)$.

As in [7], we shall deduce Theorem 1 from a slightly more general result concerning completely multiplicative functions whose values are kth roots of unity. Let F_k denote the set of all such functions; that is,

$$F_k = \{f : \mathbf{N}\to\mathbf{C} : f^k \equiv 1, f(nm)=f(n)f(m)\ (n,m\in\mathbf{N})\}.$$

THEOREM 2. *Let k be a positive integer. There exists a constant $c_0(k)$ such that for any function $f\in F_k$ there exists a positive integer $n\leq c_0(k)$ with $f(n)=f(n+1)=1$.*

The deduction of Theorem 1 from Theorem 2 is easy and will be given at the end of this section. The proof of Theorem 2 is based on the same ideas as

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