# On Conformal Welding and Quasicircles

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## 1. Introduction

Let $C$ be a quasicircle (i.e., the image of a circle under a quasiconformal mapping) and let $G_0, G_\infty$ be the bounded and unbounded components of $\hat{\mathbf{C}} \setminus C$. Throughout this paper we will assume that $0 \in G_0$. By $\omega_0, \omega_\infty$ we denote the harmonic measures on $C$, evaluated at $0, \infty$. We consider the conformal mappings $f: \mathbf{D} \to G_0$, $f(0) = 0$ and $g: \mathbf{D} \to G_\infty$, $g(0) = \infty$, where $\mathbf{D}$ is the unit disc $\{|z| < 1\}$. The welding $\varphi: \mathbf{T} \to \mathbf{T}$ is defined by

$$\varphi(\zeta) = (g^{-1} \circ f)(\zeta), \quad \zeta \in \mathbf{T}, \tag{1}$$

where $\mathbf{T}$ is the unit circle $\{|z| = 1\}$. Since $C$ is a quasicircle, the welding $\varphi$ is quasisymmetric.

We are interested in quasicircles $C$ that are "far away from being smooth." For $w_1, w_2 \in C$ let $\langle w_1, w_2 \rangle$ denote the smaller subarc of $C$ with endpoints $w_1, w_2$. We define

$$\beta(C) = \inf_{w_1, w_2 \in C} \sup_{w \in \langle w_1, w_2 \rangle} \frac{|w_1 - w| + |w_2 - w|}{|w_1 - w_2|}. \tag{2}$$

Clearly $\beta(C) \geq 1$, and since $C$ is a quasicircle the right-hand side of (2) remains bounded if we replace inf by sup. If $C$ has a tangent at some point $w \in C$, then $\beta(C) = 1$. Of course there are quasicircles $C$ with $\beta(C) > 1$, for example the snowflake. Other examples are given in Section 3.

We will use the abbreviation dim for Hausdorff dimension.

THEOREM. *Let $C$ be a quasicircle with $\beta(C) > 1$. Then there is a set $E \subset \mathbf{T}$ with*

$$\dim E < 1 \quad \textit{and} \quad \dim \varphi(\mathbf{T} \setminus E) < 1. \tag{3}$$

Tukia [11] recently constructed quasisymmetric mappings $\varphi$ satisfying (3). With the theorem we get a new class of examples.

The proof of the theorem relies on the following proposition.

PROPOSITION. *For any quasicircle $C$ there are positive constants $c, \epsilon_0$ and a number $\delta \geq 0$, where $\delta$ depends only on $\beta(C)$, such that the following*

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