Traces and the Bass Conjecture

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1. Introduction

Let A be an arbitrary ring with unit. If P is a finitely generated projective A-module then one would like to associate to P a rank function generalizing the function which assigns to the free A-module A^n the integer n. Since in the commutative case n is the trace of the identity endomorphism of A^n, one wishes to define a trace for endomorphisms of finitely generated projective A-modules in the case of noncommutative A. This was achieved independently by Hattori [7] and Stallings [12]. Unfortunately, in order for the "trace" to have the natural property of a trace function (i.e., for the trace of $a \cdot b$ to be equal to the trace of $b \cdot a$), one is forced to have the trace take values not in A but in $A/[A,A]$, where $[A,A]$ is the subgroup of A generated by all commutators $ab-ba$. The resulting trace function

$$\mathrm{tr}_P \colon \mathrm{End}_A(P) \to A/[A,A]$$

has many of the properties of the trace function in the commutative case, including additivity, commutativity, and linearity. For details, see [2]. For the results of this paper, the only two properties that will be needed are as follows.

(1) *Functoriality.* If $\alpha: A \to B$ then α induces a map $\alpha: A/[A,A] \to B/[B,B]$, and if $u \in \mathrm{End}_A(P)$ then

$$\mathrm{tr}_{P\otimes B}(u \otimes \mathrm{id}) = \alpha_*(\mathrm{tr}_P(u)).$$

(2) *Linearity.* Suppose $P = P_1 \oplus P_2$ and $u \in \mathrm{End}_A(P)$ restricts to $u_1 \in \mathrm{End}_A(P_1)$ and to $u_2 \in \mathrm{End}_A(P_2)$; then

$$\mathrm{tr}_P(u) = \mathrm{tr}_{P_1}(u_1) + \mathrm{tr}_{P_2}(u_2).$$

This last property allows one to note that, if P is a finitely generated projective A-module and one defines the rank r_P of P to be $\mathrm{tr}_P(\mathrm{id}_P)$, then if P is a direct summand of the free A-module F and $e: F \to F$ is the idempotent defining P (i.e., $P = e(F)$) then $r_P = \mathrm{tr}_F(e)$. Also, since $e \in M_d(A)$ for some d and the matrix defining e only involves finitely many elements of A, we see

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