Biholomorphic Transformations That Do Not Extend Continuously to the Boundary

B. L. FRIDMAN

Let $f: D_1 \to D_2$ be a biholomorphic transformation between bounded domains in a complex manifold. The general question considered by a number of authors (see, e.g., [1–11; 14; 16]) is whether f admits a continuous or even a smooth extension to the boundary ∂D_1 of D_1. Most of the known results are positive; that is, if D_1, D_2 are some special domains (strictly pseudoconvex, analytic polyhedra, etc.; see [3; 15] for review) then f can be extended to the boundary to provide homeomorphism or diffeomorphism (in case of smooth boundaries) between $\bar{D}_1$ and $\bar{D}_2$. Counterexamples are hard to come by, probably because of the rigidity of biholomorphic mappings. Only a few are known at this time (see [1; 2; 11]).

The purpose of this paper is to present several more negative results in $\mathbf{C}^n$, $n > 1$. We provide two constructions based on a new idea.

Both of our theorems below provide counterexamples to the general question of whether a biholomorphism $f: D_1 \to D_2$ can be extended continuously to the boundary. The first theorem gives an example of domains with topologically complicated boundaries, whereas the second theorem deals with a more regular case.

In the theorem below it is assumed that $\mathbf{C}^n \subset \mathbf{C}^{n+1}$ in a natural way: $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ is identified with $(z_1, \ldots, z_n, 0) \in \mathbf{C}^{n+1}$.

THEOREM 1. *Let G_1, G_2 be bounded domains in $\mathbf{C}^n$. Then there are bounded domains $D_1, D_2 \subset \mathbf{C}^{n+1}$ such that:*

(1) *$G_i \subset \partial D_i$, $i = 1, 2$.*
(2) *There is a biholomorphic transformation $f: D_1 \to D_2$ such that f can be extended as homeomorphism to $F: (\bar{D}_1 \backslash \bar{G}_1) \to (\bar{D}_2 \backslash \bar{G}_2)$. Moreover, f can be extended to a neighborhood of $\bar{D}_1 \backslash \bar{G}_1$ as a biholomorphic transformation.*

REMARK 1.1. Clearly, since G_1, G_2 are arbitrary, f in most cases cannot be extended to the boundary as homeomorphism. From the construction that follows one will see that f cannot be extended continuously to $\bar{G}_1$ in any case.

Received September 21, 1989. Revision received June 27, 1990.
Michigan Math. J. 38 (1991).