Invariant Diagonal Operator Ranges

BEN MATHES

Introduction

It is a well-known fact that a subspace M of a Hilbert space $\mathcal{H}$ is invariant under a set $\mathcal{S}$ of operators on $\mathcal{H}$ if and only if $M^\perp$ is invariant under the set $\mathcal{S}^* \equiv \{T^*: T \in \mathcal{S}\}$. There is no similar statement for operator ranges. Indeed, if $\mathcal{A}$ is the algebra of operators that are lower triangular relative to an orthonormal basis $\mathcal{E} = \{e_0, e_1, e_2, \ldots\}$ of $\mathcal{H}$, and if $\text{Lat}_{1/2}\,\mathcal{A}$ denotes the lattice of operator ranges invariant under $\mathcal{A}$, then

$$\text{Lat}_{1/2}\,\mathcal{A} = \text{Lat}\,\mathcal{A} \equiv \{M: M \text{ a closed invariant subspace of } \mathcal{A}\}.$$

On the other hand, $\text{Lat}_{1/2}\,\mathcal{A}^*$ properly contains $\text{Lat}\,\mathcal{A}^*$ (for the proofs of these assertions, we refer the reader to [1], [2], and [9]). All of the invariant ranges of these algebras may be obtained as ranges of diagonal operators. The purpose of this paper is to replace $\mathcal{A}$ with small subalgebras, the commutants of certain strictly cyclic weighted shifts, and then characterize the ranges of diagonal operators invariant under these smaller algebras. In a paper to appear as a sequel to the one in hand, we will investigate the ranges of diagonal operators that are invariant under the adjoints of these smaller algebras. Our results suggest that the difference between the ranges of diagonal operators invariant under these smaller algebras and those invariant under their adjoints is the same difference seen when passing from $\mathcal{A}$ to $\mathcal{A}^*$.

Preliminaries

Assume for the moment that $\mathcal{A}$ is the commutant of the unilateral shift operator S, that is, the operator defined by $Se_i = e_{i+1}$ $(i = 0, 1, \ldots)$. We assert that there are no nontrivial invariant ranges of diagonal operators under $\mathcal{A}$. By nontrivial, we mean ranges other than the obvious invariant closed subspaces of all lower triangular operators. This may be seen by assuming that $D = \text{diag}(d_i)$ is invariant under $\mathcal{A}$, then proving that there exists $m \geq 0$ and $\epsilon > 0$ such that $d_i = 0$ for all $0 \leq i < m$, and $d_i \geq \epsilon$ whenever $i \geq m$ (there is no

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