Bounds for the Fundamental Frequencies of an Elastic Medium

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I. Introduction

Let Ω be an open bounded region in $\mathbf{R}^n$. Let Δ be the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

on real-valued functions on Ω. Let $\boldsymbol{\Delta}$ be the Laplace operator applied to components of $\mathbf{R}^n$-valued functions on Ω. Let $\partial\Omega$, $\partial/\partial n$, grad, and div be (respectively) the boundary set of Ω, the outward-pointing normal derivative on $\partial\Omega$, the gradient, and the divergence. Let $L^2(\Omega)$ be the Hilbert space of square-integrable real-valued functions on Ω and let $\mathbf{L}^2(\Omega)$ be the Hilbert space of $\mathbf{R}^n$-valued functions $\mathbf{u}$ on Ω, so that the pointwise $\mathbf{R}^n$ norm $\|\mathbf{u}(x)\|$ is in $L^2(\Omega)$.

Consider the three eigenvalue problems:

$$\boldsymbol{\Delta}\mathbf{u} + \alpha\,\mathrm{grad}(\mathrm{div}\,\mathbf{u}) + \Lambda^{(\alpha)}\mathbf{u} = \mathbf{0} \text{ in } \Omega, \qquad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega; \tag{1.1}$$

$$\Delta v + \lambda v = 0 \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega; \qquad \text{and} \tag{1.2}$$

$$\Delta^2\phi + \nu\Delta\phi = 0 \text{ in } \Omega, \qquad \phi = \frac{\partial\phi}{\partial n} = 0 \text{ on } \partial\Omega. \tag{1.3}$$

The number α is a nonnegative constant, and in this paper α always refers to the constant in problem (1.1). We consider the collection of eigenvalue problems in (1.1) for all nonnegative values of α. Problem (1.1) governs the behavior of an elastic medium and thus appears often in the theory of elasticity. The problems (1.2) and (1.3) are often referred to as the Dirichlet and buckling eigenvalue problems, respectively.

In this paper we extend some inequalities obtained recently by Kawohl and Sweers [3] for the smallest eigenvalue of problem (1.1) to all of its eigenvalues. We assume that the eigenvalues of (1.1), (1.2), and (1.3) are ordered $\Lambda_1^{(\alpha)} \le \Lambda_2^{(\alpha)} \le \cdots$, $\lambda_1 \le \lambda_2 \le \cdots$, and $\nu_1 \le \nu_2 \le \cdots$, respectively.

We establish that the following inequalities hold:

$$\lambda_{[(n+k-1)/n]} \le \Lambda_k^{(\alpha)} \quad \text{for all } \alpha \ge 0, \text{ all } k, n, \tag{1.4}$$

Received March 1, 1988. Final revision received September 4, 1990.
Michigan Math. J. 38 (1991).