On the Quasidiagonality of Direct Sums of Normal Operators and Shifts

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1. Introduction

1.0. Let $\mathcal{H}$ be a complex, infinite-dimensional, separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators acting on $\mathcal{H}$. By $\mathcal{K}(\mathcal{H})$ we denote the compact operators, and $\mathcal{A}(\mathcal{H})=\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra. The canonical map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{A}(\mathcal{H})$ is denoted π, and for $T\in\mathcal{B}(\mathcal{H})$ the spectrum of T is written $\sigma(T)$; $\sigma_e(T)=\sigma(\pi(T))$ will denote the essential spectrum of T.

An operator $T\in\mathcal{B}(\mathcal{H})$ is said to belong to the class (QD) of *quasidiagonal* operators if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of finite-dimensional projections converging strongly to the identity operator I for which $\{\|TP_n-P_nT\|\}_{n=1}^{\infty}$ converges to 0. If $TP_n=P_nT$ for all $n\geq 1$, then T is said to be *block-diagonal* and we write $T\in(BD)$. It is well known that (QD) is closed, invariant under compact perturbations, and given $\epsilon>0$ and $T\in(QD)$, there exist $B\in(BD)$ and $K\in\mathcal{K}(\mathcal{H})$, $\|K\|<\epsilon$, such that $T=B+K$.

An operator T is called *quasitriangular* ($T\in(QT)$) if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ as above for which $\lim_{n\to\infty}\|(I-P_n)TP_n\|=0$, and T is *bi-quasitriangular* ($T\in(BQT)$) if both T and T^* are quasitriangular. Both (QT) and (BQT) have been studied extensively (cf. [12] and its references) and much is known about these classes. It is not hard to see that $(QD)\subseteq(BQT)$ and $T\in(QD)$ if and only if there is a sequence $\{P_n\}_{n=1}^{\infty}$ as above which simultaneously implements the quasitriangularity of T and T^*.

1.1. Recall that $W\in\mathcal{B}(\mathcal{H})$ is called a *bilateral* (resp. *unilateral*) weighted shift if there exists an orthonormal basis $\{e_n\}_{n\in\mathbf{Z}}$ (resp. $\{e_n\}_{n\geq 1}$) of $\mathcal{H}$ and a bounded sequence of scalars $\{\omega_n\}_{n\in\mathbf{Z}}$ (resp. $\{\omega_n\}_{n\geq 1}$) such that $We_n=\omega_n e_{n+1}$ for all n. Up to unitary equivalence (under which quasidiagonality is invariant), we may and do assume that all weights are positive. The bilateral (resp. unilateral) shift with all weights equal to 1 is denoted by B (resp. S).

If W is a bilateral weighted shift, then we say that W is *block-balanced* if, given $\epsilon>0$ and $n>0$ an integer, there exist integers p and q such that $p+n<0<q$ and

$$\|(\omega_p,\omega_{p+1},\omega_{p+2},\dots,\omega_{p+n})-(\omega_q,\omega_{q+1},\omega_{q+2},\dots,\omega_{q+n})\|_\infty<\epsilon.$$

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