A Hypercontractive Estimate for the Heat Semigroup on $\mathbf{P}^2$

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1. Introduction

Hypercontractive estimates for diffusion semigroups first appeared in the work of Nelson [7] in connection with certain problems in constructive quantum field theory. Perhaps the most important tool used to obtain this type of estimate is the logarithmic Sobolev inequality, introduced by Gross in [4]. Such inequalities can be viewed as limiting cases of Sobolev inequalities, and that is the approach we take in this work.

Denote by $\mathbf{P}^n$ n-dimensional real projective space, which we will view as the n-sphere S^n with antipodal points identified. We take $d\zeta$ to be normalized Lebesgue measure on either $\mathbf{P}^n$ or S^n. Functions on $\mathbf{P}^n$ will then be naturally identified with even functions on S^n. For such a function f we will write

$$\|f\|_{L^p(\mathbf{P}^n)}=\left(\int_{\mathbf{P}^n}|f|^p\,d\zeta\right)^{1/p}=\left(\int_{S^n}|f|^p\,d\zeta\right)^{1/p}=\|f\|_{L^p(S^n)}.$$

Using an appropriate logarithmic Sobolev inequality, Mueller and Weissler [6] obtained the following hypercontractive estimate for the heat semigroup on S^n: for $1\le p\le q<\infty$ and $t\ge(1/2^n)\ln[(q-1)/(p-1)]$,

$$\|e^{t\Delta}f\|_{L^q(S^n)}\le\|f\|_{L^p(S^n)}. \tag{1.1}$$

Subsequently, Bakry and Emery [1] gave an alternative proof in terms of second-order estimates for the semigroup generator and Beckner [2] proved the stronger result that the Poisson semigroup is hypercontractive.

From the previous discussion, we see that the estimate (1.1) will also hold for functions on $\mathbf{P}^n$. A natural question to ask is whether or not any improvement on this estimate is possible for $\mathbf{P}^n$. The purpose of this note is to prove the following theorem which gives such an improvement on $\mathbf{P}^2$.

MAIN THEOREM (THEOREM 3.1). *Let $f\in L^p(\mathbf{P}^2)$. Then for $1\le p\le q<\infty$ and $t\ge\frac{1}{8}\ln[(q-1)/(p-1)]$,*

$$\|e^{t\Delta}f\|_{L^q(\mathbf{P}^2)}\le\|f\|_{L^p(\mathbf{P}^2)}.$$

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