A Probabilistic Zero Set Condition for the Bergman Space

EMILE LEBLANC

Introduction

A function f, analytic in the open unit disk $\mathbf{D}$, is said to belong to a Bergman space L_a^p, $0<p<\infty$, if

$$\int_{\mathbf{D}} |f(z)|^p \, dA(z) < \infty,$$

where $dA(z)$ is area measure on $\mathbf{D}$. (The space L_a^2 is referred to as the Bergman space, and L_a^∞ is defined to be H^∞.)

Axler [1] gives a short introduction to the Bergman spaces with proofs of the basic facts about these spaces; however, describing the zero sets of the functions in the Bergman spaces remains an unsolved problem.

This paper presents a condition on a sequence $r_1, r_2, \ldots \in [0,1]$ that is weaker than the Blaschke condition, namely,

$$\limsup_{\epsilon \to \infty} \frac{\sum_{j=1}^{\infty}(1-r_j)^{1+\epsilon}}{\log(1/\epsilon)} < \frac{1}{4},$$

that guarantees that a set of points in the disk with moduli r_j and random arguments is almost surely the zero set of a function in L_a^2. An explicit construction of a function with the desired zero set that almost surely belongs to the Bergman space is provided (using Horowitz's generalization of the Blaschke factors).

It is well known (see, e.g., [5, pp. 90–95]) that a countable set $S=\{z_j\}$ of points (assumed to be ordered by magnitude) in $\mathbf{D}$ is a zero set for an H^p function, $0<p\le\infty$, if and only if the points satisfy the Blaschke condition:

$$\sum_{z_j \in S} (1-|z_j|) < \infty.$$

No such simple condition for the zero sets for L_a^p functions is known. Horowitz obtained many interesting results about zero sets in the Bergman spaces in [4]. There are three results in particular that highlight the differences and similarities between the Hardy spaces and the Bergman spaces:

Received August 3, 1989. Revision received June 6, 1990.
Michigan Math. J. 37 (1990).