A Monotone Slit Mapping with Large Logarithmic Derivative

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1. Introduction

We denote the unit disc in the complex plane, that is $\{z : |z| < 1\}$, by Δ. The class $\mathcal{S}$ is the class of all functions $f(z)$ which are analytic and univalent in Δ and for which $f(0) = 0$ and $f'(0) = 1$. If $f(z)$ is in $\mathcal{S}$ we set

$$I_2\left(r, \frac{f'}{f}\right) = \int_0^{2\pi} \left| \frac{rf'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta,$$

for each r with $0 < r < 1$. By a monotone slit mapping we mean a function $f(z)$ in $\mathcal{S}$ whose image domain is the complement of a path $\Gamma(t)$ on $[0, \infty)$ for which $|\Gamma(t_1)| < |\Gamma(t_2)|$ if $t_1 < t_2$. That is, Γ meets each circle centred on the origin at most once.

In this note we prove the following.

THEOREM 1. *There is a monotone slit mapping* $\mathfrak{F}(z)$ *for which*

$$I_2\left(r, \frac{\mathfrak{F}'}{\mathfrak{F}}\right) \neq o\left(\frac{1}{1-r} \log\log \frac{1}{1-r}\right) \tag{1.1}$$

as $r \to 1$.

A standard way to obtain information on logarithmic coefficients of a function f in $\mathcal{S}$ is to estimate $I_2(r, f'/f)$ (cf. [7]). The logarithmic coefficients play an essential role, for example in the proof of the Bieberbach conjecture by de Branges [5].

Our starting point is an estimate of Biernacki in [4] (see also [11, p. 151]) that if f is a function in $\mathcal{S}$ then, as $r \to 1$,

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right).$$

It is surprising, but nevertheless true, that this elementary bound is best possible. Hayman produced in [11] an example of a function $f(z)$ in $\mathcal{S}$ for which

$$I_2\left(r, \frac{f'}{f}\right) \neq o\left(\frac{1}{1-r} \log \frac{1}{1-r}\right).$$

Our construction borrows much from the methods he employed there.

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