Discrete Quasiconformal Groups with Small Dilatation

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1. Introduction

In this paper we consider discrete uniformly quasiconformal groups which act on $\bar{\mathbf{R}}^n$, $n \geq 3$. We provide examples of such groups which have small dilatation, yet are not quasiconformally conjugate to conformal (i.e., Möbius groups) on $\bar{\mathbf{R}}^n$. Examples of both elementary discrete groups (limit set of at most two points) and non-elementary (an uncountable, perfect limit set) discrete groups are furnished.

A natural way to construct a quasiconformal group acting on $\bar{\mathbf{R}}^n$ is to conjugate a conformal group by a quasiconformal mapping. Indeed, Gehring and Palka [5] first raised the question whether every uniformly quasiconformal group might not be of this form. This question was answered in the affirmative by Sullivan [10] and Tukia [11] for groups acting on subsets of $\bar{\mathbf{R}}^2$. Hinkkanen [6; 7] has shown that if G is a quasisymmetric (i.e., a 1-dimensional quasiconformal) group G acting on $\mathbf{R}$, then there is a quasisymmetric function f such that $f^{-1} \circ G \circ f$ is a group of linear functions. Later, Tukia [12] constructed for each $n \geq 3$ a quasiconformal group which is not isomorphic to, and hence not quasiconformally conjugate to, a Möbius group. Methods used by Tukia were later modified by Martin [8] to yield discrete quasiconformal groups which are not quasiconformally conjugate to a Möbius group. Further examples were provided by Gehring and Martin [3] as well as by Freedman and Skora [2]. As each of these groups possesses a large dilatation, we asked whether a uniformly quasiconformal group with sufficiently small dilatation must be quasiconformally conjugate to a conformal group. The answer is no; by a modification of Tukia's methods we showed in [9] that for each $n \geq 3$ and $K > 1$ there is a K-quasiconformal group acting on $\bar{\mathbf{R}}^n$ which is not the quasiconformal conjugate of a Möbius group. Our examples, like Tukia's, were not discrete, so the question remained whether such examples existed in the category of discrete quasiconformal groups. While the methods used by Martin to extract elementary discrete subgroups from Tukia's group are entirely applicable to our groups of small dilatation, the modifications he used to obtain non-elementary discrete