Moduli Spaces of Polynomial Minimal Immersions between Complex Projective Spaces

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1. Introduction and Preliminaries

In [1; 6] Do Carmo and Wallach showed that the space of full k-homogeneous polynomial minimal immersions of the m-sphere S^m into any n-sphere S^n (for various n) can be parametrized by a compact convex body lying in a finite-dimensional vector space. They also gave a lower bound for the dimension in terms of m and k. The objective of this note is to construct Do Carmo–Wallach type moduli spaces of (homotopically nontrivial) minimal immersions between complex projective spaces. More precisely, for $m \geq 2$ and $p > q \geq 0$, we consider $\mathcal{H}^{p,q} = \mathcal{H}^{p,q}_{m+1}$, the complex vector space of harmonic polynomials on $\mathbf{C}^{m+1}$ of degree p in $z_0, \dots, z_m \in \mathbf{C}$ and degree q in $\bar{z}_0, \dots, \bar{z}_m \in \mathbf{C}$. An element of $\mathcal{H}^{p,q}$ is completely determined by its restriction to the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$. A map $f: S^{2m+1} \to S^{2n+1}$ between the unit spheres of $\mathbf{C}^{m+1}$ and $\mathbf{C}^{n+1}$ is said to be a *polynomial map of bidegree* (p,q) if the coordinates of f belong to $\mathcal{H}^{p,q}$. In this case, as $\mathcal{H}^{p,q}$ consists of (complex-valued) spherical harmonics, f is a harmonic map in the sense of Eells and Sampson [2; 3]. There are three immediate consequences of homogeneity:

(1) f factors through the Hopf bundle maps $\pi: S^{2m+1} \to \mathbf{C}P^m$ and $\pi: S^{2n+1} \to \mathbf{C}P^n$ inducing a map $F: \mathbf{C}P^m \to \mathbf{C}P^n$;

(2) F pulls back the canonical line bundle of $\mathbf{C}P^n$ to the $(p-q)$th power of that of $\mathbf{C}P^m$, in particular, F has degree $p-q > 0$ (on second cohomology) and, consequently, $m \leq n$;

(3) the induced map $F: \mathbf{C}P^m \to \mathbf{C}P^n$ is harmonic if and only if f is *horizontal* with respect to the Hopf fibrations (i.e., if the differential of f maps $(\ker \pi_*)^\perp \subset T(S^{2m+1})$ into $(\ker \pi_*)^\perp \subset T(S^{2n+1})$). (This follows from the reduction theorem of Smith [2].)

If, in addition to f being horizontal, F is homothetic then it is minimal [2], and we call $F: \mathbf{C}P^m \to \mathbf{C}P^n$ *the polynomial minimal immersion of bidegree* (p,q) induced by f. To formulate our main result we recall that a map $F: \mathbf{C}P^m \to \mathbf{C}P^n$ is said to be *full* if the image of F is not contained in a

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