# Support Sets and Gleason Parts

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## 1. Introduction

The function algebra $H^\infty$ is the collection of all bounded holomorphic functions in the unit disc $D$ of the complex plane. Under the supremum norm it is a Banach algebra. The Gelfand theory represents $H^\infty$ as a subalgebra of $C(M)$, the algebra of continuous, complex-valued functions on $M$, the maximal ideal space of $H^\infty$. With the weak-star topology $M$ is a compact Hausdorff space, and the point evaluations for points in the disc form a dense subset [3].

For points $z$ and $w$ in the disc, the pseudo-hyperbolic distance from $z$ to $w$ is

$$\rho(z,w)=\left|\frac{z-w}{1-\bar{w}z}\right|.$$

Pick's lemma states that, for $z$ and $w$ in $D$ and $f$ a nonconstant $H^\infty$ function with norm not exceeding 1, $\rho(f(w),f(z))\le\rho(z,w)$. Taking points $\phi$ and $\psi$ in $M$ and extending $\rho$ to $M\times M$ by $\rho(\phi,\psi)=\sup\{\rho(f(\phi),f(\psi)):f\in H^\infty,\ \|f\|_\infty<1\}$, we can partition $M$ into equivalence classes known as Gleason parts, calling $\phi$ and $\psi$ equivalent provided $\rho(\phi,\psi)<1$. We denote the Gleason part to which $\phi$ belongs by $P_\phi$.

Hoffman [9] has shown that the Gleason parts of $M$ are either singletons or discs. For the latter case he constructed [11] a one-to-one map $L_m$ of $D$ onto $P_m$ sending 0 to $m$ such that $f\circ L_m$ is holomorphic for all $f$ in $H^\infty$. Such parts and points are called *analytic,* while points whose Gleason parts are singletons are called *trivial.*

Viewing $H^\infty$ functions as continuous over the Shilov boundary of $M$, which is the maximal ideal space of $L^\infty$ and which we denote by $X$, one can represent an element $\phi$ of $M$ as integration against a positive measure $\mu_\phi$: $f(\phi)=\int f\,d\mu_\phi$. This representation allows us to extend $\phi$ to $L^\infty$ in such a manner that the Gelfand transforms of $L^\infty$ functions are also continuous on $M$. The measure $\mu_\phi$ is called a *representing measure,* and its support in $X$ is known as a *support set*. Points in the same Gleason part have the same support set [9]. Support sets may meet, but if they do then one is entirely contained within the other (unpublished work of Hoffman).

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