Complex Analytic Curves of Minimal Area in Cubes

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1. Introduction

In this paper we prove that, in a cube in $\mathbf{C}^n$, any complex analytic curve passing through the center of this cube has the area which is not less than the area of a complex line containing the center. For the proof, we find the minimal length of real curves, lying on the boundary of the cube and intersecting each real hyperplane passing through the center, in at least m points.

It is well known that the minimal volume of the intersection of the ball in $\mathbf{C}^n$ with center at the origin, and a k-dimensional analytic set passing through the origin, is equal to the volume of a k-dimensional plane with the same property. This statement is not true for an arbitrary symmetric convex domain. The corresponding examples were constructed in [2] and [4].

It is important for some applications to know the minimal value of volume of analytic hypersurface passing through the origin in cubes, since $\mathbf{C}^n$ can be packed by cubes without holes and intersections. Such packing was used for example in [3] to get conditions for the uniqueness of entire functions of exponential type vanishing on some discrete set in $\mathbf{R}^n$. The exact conditions were obtained in this paper only when $n=2$ since in [1] it was proved that, for cubes in $\mathbf{C}^2$, the area minimizing analytic curves containing the cube's center are plane sections.

The proof of the theorem in [1] was based on two facts. First, it was noted that the intersection of a real hyperplane and an analytic curve contains two distinct points of the boundary of the cube if the hyperplane and the curve contain the origin. Second, the area of the curve was expressed through the length of its intersections with cubes of smaller diameter. Therefore, the problem was reduced to a completely real case: to find the minimal length of curves on the boundary of cubes if the intersection of a curve and any hyperplane section passing through the origin contains at least k points. It was proved in [1] that for cubes in $\mathbf{R}^3$ and $\mathbf{R}^4$ real curves of minimal length are intersections of planes, parallel to cube's sides, with the boundary of the cube.

In our paper we prove that the solution of the real problem is the same if the cube has arbitrary dimension and, therefore, we get the exact lower

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