Some Results about the Space $A^{-\infty}$ of Analytic Functions

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Introduction and Results

In this paper, we prove some results about the class $A^{-\infty}$ of analytic functions on the unit disc $D=\{z\in\mathbf{C}:|z|<1\}$ that satisfy $|f(z)|\le C(1-|z|)^{-n}$ for some $C>0$ and $n\in\mathbf{N}$. This class was studied extensively by Korenblum in [6], where some results about the moduli of the zeros of the functions in the space $A^{-\infty}$ are given. There ([6, p. 202], see also [8, Thm. 6]), a function in $A^{-\infty}$ is constructed whose sequence of zeros $(z_n)_n$ satisfies $\sum_n(1-|z_n|)=+\infty$; so, in general, the Blaschke product cannot be defined. We shall prove that the function $f\in H(D)$, defined by $f(z)=g_3(\tau)$ where $z=e^{2\pi i\tau}$, $\Im\tau>0$ and g_3 is the well-known Eisenstein invariant (see [1, p. 12]), belongs to $A^{-\infty}$ and f also satisfies $\sum_n(1-|a_n|)=+\infty$, where $(a_n)_n$ is its sequence of zeros in D.

It is easy to prove (see [8, p. 224]) that the function

(1) $$f(z)=\sum_{n\ge0}a_nz^n \quad \text{belongs to } A^{-\infty} \text{ if and only if } (a_n)_n\in s',$$

where s′ is the space of tempered sequences in which $(a_n)_n\in s'$ if there exist C and $\alpha>0$ such that $|a_n|\le C(n+1)^\alpha$. So the boundary values of the functions of the space $A^{-\infty}$ are the distributions on the circle $\mathbf{T}=\{z\in\mathbf{C}:|z|=1\}$ with vanishing negative Fourier coefficients. Moreover, if $f(z)=\sum_{n\ge0}a_nz^n$ and $u\in D'(\mathbf{T})$ is its boundary value, then $a_n=\langle u,e^{-in\theta}\rangle$.

In the following theorem, we give an analogous identification for the functions in the space $A^{-\infty}$ as some Fourier–Laplace type transforms of the tempered distributions with support contained in $[0,+\infty)$.

THEOREM 1. *A function f belongs to the space* $A^{-\infty}$ *if and only if there exists a tempered distribution* u_f *with* $\operatorname{supp}(u_f)\subset[0,+\infty)$ *such that* $f(z)=\langle u_f(t),e^{t(z+1)/(2z-2)}\rangle$ *if* $|z|<1$. *Moreover, if* $f(z)/(1-z)=\sum_n a_nz^n$ *then* $a_n=\langle u_f,L_n(t)e^{-(t/2)}\rangle$, *where the* $L_n(t)$ *are Laguerre polynomials.*

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