On the Functional Calculus of Contractions with Nonvanishing Unitary Asymptotes

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1. Introduction

Let T be an absolutely continuous contraction on the (complex, separable) Hilbert space $\mathcal{H}$. The Sz.-Nagy–Foiaş functional calculus for T is a contractive, weak*-weak*-continuous algebra homomorphism from the Hardy space H^∞ into the set of all (bounded, linear) operators defined on $\mathcal{H}$:

$$\phi_T: H^\infty \to \mathcal{B}(\mathcal{H}), \qquad \phi_T: h \mapsto h(T)$$

(see [21]). We recall that the spaces H^∞ and $\mathcal{B}(\mathcal{H})$ can be identified as the duals of the factor space L^1/H_0^1 and the von Neumann–Schatten ideal $\mathcal{C}_1(\mathcal{H})$ of trace class operators, respectively (cf. [19]). (All function spaces are defined with respect to the normalized Lebesgue measure m on the unit circle ∂D, and H_0^1 means the set of functions with vanishing Fourier coefficients of nonpositive index in L^1.) The mapping ϕ_T induces a contractive, sesquilinear transformation $p_T: \mathcal{H} \times \mathcal{H} \to L^1/H_0^1$, where $p_T(x,y)$ is the unique element of L^1/H_0^1 possessing the property that $\langle h(T)x, y\rangle = \int_{\partial D} hf\,dm$ holds for every $h \in H^\infty$ and every integrable function f in the coset $p_T(x,y)$. The value $p_T(x,y)$ can be considered as the "local density function" of ϕ_T at x, y.

The powerful theorem asserting that p_T is surjective if ϕ_T is an isometry was proved simultaneously and independently by Bercovici [2] and Chevreau [8]. This theorem has many consequences. For example, it follows that any absolutely continuous contraction with an isometric functional calculus has a nontrivial invariant subspace, a theorem first proved by Brown, Chevreau, and Pearcy [5]. In [6], Brown and Chevreau have shown that the contractions with isometric functional calculus are even reflexive.

Let $T^{(a)} \in \mathcal{B}(\mathcal{H}^{(a)})$ denote the unitary asymptote of the contraction T (cf. [17]). $T^{(a)}$ is an absolutely continuous unitary operator, and $\mathcal{H}^{(a)} \neq \{0\}$ if and only if $\lim_{n\to\infty}\|T^n x\| \neq 0$ for some nonzero vector $x \in \mathcal{H}$. Let $\Gamma = \Gamma_T$ be a Borel set on the unit circle ∂D such that $\chi_\Gamma\,dm$ is a scalar spectral measure for $T^{(a)}$. (χ_Γ stands for the characteristic function of Γ.) It is not difficult to show (see Lemma 3) that if $m(\Gamma_T) = 1$ then ϕ_T is an isometry and so the Bercovici–Chevreau theorem applies. Thus $m(\Gamma_T) = 1$ implies that $\operatorname{ran} p_T = L^1/H_0^1 = \pi(L^1(\Gamma_T))$. (Here $\pi: L^1 \to L^1/H_0^1$; $\pi: f \mapsto [f]$ denotes the factor map-

Received December 21, 1988. Revision received February 20, 1989.
Michigan Math. J. 37 (1990).