On the Coefficients of the Mapping to the Exterior of the Mandelbrot Set

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Introduction

The Mandelbrot set M arises in the dynamics of complex quadratic polynomials $q_w(z)=z^2+w$. It consists of those parameter values w such that the Julia set obtained from q_w is connected. The complement $\tilde{M}$ of M in the Riemann sphere is known [1] to be simply connected and to have mapping radius equal to 1. Thus we may consider the analytic homeomorphism

$$\psi(z)=z+\sum_{m=0}^{\infty} b_m z^{-m} \tag{1}$$

of $\Delta=\{z: 1<|z|\leq\infty\}$ onto $\tilde{M}$. It is our purpose to give a useful formula for the coefficients b_m and to show that many of these coefficients are zero. We also determine infinitely many nonzero coefficients, and conclude with a description of the Faber polynomials of the Mandelbrot set. Our work is motivated by an article by Jungreis [2].

Background

Define recursively the polynomials $p_1(w)=w^2+w$ and

$$p_n(w)=p_{n-1}(w)^2+w \tag{2}$$

for $n>1$. Evidently, p_n is a monic polynomial of degree 2^n. It is known [2] that the zeros of p_n all lie in M, and so it is possible to define in $\tilde{M}$ a single-valued branch of $p_n(w)^{1/2^n}=w+O(1)$ as $w\to\infty$. In what follows we shall also have use for $p_n^{m/2^n}\equiv[p_n^{1/2^n}]^m$ for positive integers m.

It turns out [2] that the functions $\phi_n\equiv p_n^{1/2^n}$ converge locally uniformly in $\tilde{M}$ to $\phi=\psi^{-1}$, the inverse of the mapping ψ. Near ∞ the functions ϕ_n are one-to-one, and their inverse functions $\psi_n\equiv\phi_n^{-1}$ evidently satisfy

$$p_n(\psi_n(z))=z^{2^n}. \tag{3}$$

The functions ψ_n are defined in larger and larger subsets of Δ as $n\to\infty$, and converge locally uniformly in Δ to ψ. In fact, the following lemma shows that this convergence is remarkably strong.

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