Limits of Strongly Irreducible Operators, and the Riesz Decomposition Theorem

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1. Introduction

Let T be a (bounded linear) operator acting on a complex, separable, infinite-dimensional Hilbert space $\mathcal{H}$ and assume that the spectrum of T, $\sigma(T)$, is not connected. The Riesz decomposition theorem says that under these circumstances $\mathcal{H}$ can be written as the algebraic sum $\mathcal{H}_1 \dotplus \mathcal{H}_2$ of two nontrivial invariant subspaces of T; equivalently, T commutes with a nontrivial idempotent operator E. Furthermore, $E = E(T)$ can be written as a certain contour integral, and the upper semicontinuity of separate parts of the spectrum implies that every operator T' close enough to T commutes with a nontrivial idempotent $E' = E(T')$. Moreover, if T has the above property then the same is true for every operator WTW^{-1} similar to T, because $\sigma(WTW^{-1}) = \sigma(T)$.

On the other hand, in [6] Gilfeather considered the class of all strongly irreducible operators defined by

$$\mathcal{SI}(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \colon T \text{ does not commute with any nontrivial idempotent}\}.$$

(Here $\mathcal{L}(\mathcal{H})$ denotes the algebra of all operators acting on $\mathcal{H}$.)

In this note we characterize the norm-closure $\mathcal{SI}(\mathcal{H})^-$ of the class $\mathcal{SI}(\mathcal{H})$. In a certain sense, this characterization can be considered as an "approximate inverse" of the Riesz decomposition theorem. Indeed, we have the following.

THEOREM.

$$\mathcal{SI}(\mathcal{H})^- = \{T \in \mathcal{L}(\mathcal{H}) \colon \sigma(T) \textit{ is connected}\}.$$

Our introductory paragraph indicates that $\sigma(T)$ is necessarily connected for each T in $\mathcal{SI}(\mathcal{H})^-$; moreover, the class $\mathcal{SI}(\mathcal{H})$ (as well as its closure) is invariant under similarity. Thus, we must show only that every T in $\mathcal{L}(\mathcal{H})$ with a connected spectrum can be uniformly approximated by strongly irreducible operators.

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