A Picard Theorem for Projective Varieties

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In an earlier paper [3], the author proved a Picard theorem for maps of the entire plane $\mathbf{C}$ into a certain algebraic variety. The method of that paper seemed not to go to the heart of the matter, and it was obscure how far the theorem could be generalized. In the present paper we prove a more general Picard theorem by a method that seems to be the appropriate one for the problem, and we give examples to show that our result is nearly the best possible. The main element of our new method is a parametrization of a monomial variety that is familiar to students of toric varieties ([5, Introduction], [9, Prop. 1.2]). The use we make of this parametrization is of course completely different from theirs.

Let V be an irreducible algebraic variety in $\mathbf{CP}^n$, possibly singular. Let $\Pi_0, \dots, \Pi_n$ be independent hyperplanes of $\mathbf{CP}^n$. Consider a holomorphic map $f: \mathbf{C} \to V$ such that the image of f does not meet $\Pi_0, \dots, \Pi_n$. Suppose that none of the sections $V \cap \Pi_k$, $k = 0, \dots, n$, is contained in the union of the others. Then a theorem of Green [2, p. 66] (for related results see Lang [6; 7]) asserts that f is *algebraically degenerate* in the sense that its image lies in a proper hypersurface section of V.

When can this conclusion be strengthened to say that the image of f lies in a section by a hyperplane? Green gives an example [2, p. 62], similar to our Example 1, to show that this is not always so. In the present paper Theorem 2 gives a sufficient condition in terms of the intersections of V with $\Pi_0, \dots, \Pi_n$, and Theorem 3 relates this to our earlier paper [3]. The technical result needed to obtain Theorem 2 is Theorem 1. We apply the Borel lemma to show that the Zariski closure Z of $f(\mathbf{C})$ is a monomial variety and then parametrize it in the manner to which we referred above. Our result is almost that Z is a *toric variety* [9, Thm. 1.4], but the definition of a toric variety requires such a variety to be normal, which is irrelevant for our purposes. It would be possible to apply Theorem 1 to obtain refinements of Theorem 2 that referred to higher orders of contact, but in the absence of applications we have not discussed these.

When I wrote the first version of this paper I was not aware that the construction was standard in the theory of toric varieties. I am indebted to an anonymous referee for the suggestion that my argument must correspond to

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