The Volume/Diameter Ratio for Positively Curved Manifolds

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1. Statement of Main Result

In this paper, a Riemannian manifold (M^n, g) always means a connected C^∞-manifold of dimension n $(n \geq 2)$ with a Riemannian metric g. $\text{Vol}(M)$, $i(M)$, $d(M)$, K_M, and Ric_M will denote the volume, the injectivity radius, the diameter, the sectional curvature, and the Ricci curvature of M, respectively.

Gromov [4] showed that if $0 > K_M \geq -1$ $(n \geq 8)$ then

$$\text{Vol}(M) \geq C_n(1+d(M)),$$

where the constant $C_n > 0$ depends only on n. Furthermore, the Bishop volume comparison [2] gives $\text{Vol}(M) \leq C_n' d(M)$ when $\text{Ric}_M \geq n-1$. This paper is concerned with a better estimate of this type for $K_M \geq 1$. Our main result can be stated as follows.

THEOREM A. *Let M be a complete Riemannian manifold with $K_M \geq 1$; then*

$$(*) \qquad \frac{\text{Vol}(M)}{d(M)} \leq \frac{\text{Vol}(S^n)}{d(S^n)}.$$

Moreover, the equality holds if and only if M is isometric to S^n or RP^n with constant curvature $+1$.

REMARKS.

1. For $d(M) \leq \pi/2$, $(*)$ is also true when only $\text{Ric}_M \geq n-1$. This follows from the Bishop volume comparison theorem for Ricci curvature. The author does not know if the rigidity (in case equality holds) is true. However, if M is not simply connected, the rigidity is true (cf. §2).

2. For $d(M) > \pi/2$, Theorem A is wrong when only $\text{Ric}_M \geq n-1$. This can be seen using, for example, $M^4 = CP^2$ with metric normalized so that $\text{Ric}_M = 3$.

3. The interesting point is that for $K_M \geq 1$, the same conclusion holds if $d(M) > \pi/2$. For this, one can consider a pair of points at maximal distance which by Berger's lemma are mutually critical for the distance function. Then

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