Space-Preserving Composition Operators

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1. Introduction

If φ is an analytic function mapping the unit disk Δ into itself, and if f belongs to the Hardy class H^p, then the composition $(f\circ\varphi)$ belongs to H^p also. This was first pointed out by Littlewood [7]. Our object here is to consider when a reverse implication may hold. That is, let $H(\Delta)=H$ be the topological vector space of functions holomorphic on Δ and let V be a subspace of H. We ask the following question: *What are the holomorphic functions φ mapping Δ into Δ, such that whenever $f\in H$ and $(f\circ\varphi)\in V$ it follows that $f\in V$?* A function φ satisfying this condition will be said to *possess property* (*) *relative to the subspace V.*

It is immediately clear that if φ_1 and φ_2 possess property (*) then so does $(\varphi_1\circ\varphi_2)$. We will show in Example 5 of the next section that φ_1 and $(\varphi_1\circ\varphi_2)$ may possess property (*) even if φ_2 does not. As a first example, a linear fractional transformation mapping Δ onto Δ clearly possesses property (*) relative to the H^p spaces, BMOA, and the disk algebra. Further, if φ_1 is a linear fractional transformation mapping Δ onto Δ, then $(\varphi_1\circ\varphi_2)$ possesses property (*) relative to the H^p spaces, BMOA, or the disk algebra if and only if φ_2 does.

Ryff [9] proved the following theorem related to our question: *Let f be nonconstant and analytic on Δ. Let φ be analytic on Δ with $\varphi(0)=0$ and $|\varphi|<1$. Then $\|f\|_p=\|f\circ\varphi\|_p$ if and only if φ is inner.* Later, Nordgren [8] showed that *if φ is an inner function, then φ possesses property* (*) *relative to H^p. And, the composition operator C_φ is norm-preserving on H^p ($\|f\|_p=\|f\circ\varphi\|_p$) if and only if $\varphi(0)=0$.*

In this paper we introduce a family of functions φ mapping Δ into Δ for which (*) holds for the H^p spaces, BMOA, and the disk algebra. Our maps can be factored as a finite Blaschke product times a nonconstant outer function, and hence have modulus strictly less than 1 on arcs of $\partial\Delta=\mathbf{T}$ of positive measure. In addition to satisfying (*), the composition operators associated with these maps provide examples illustrating results of spectral properties of C_φ as studied by Cowen [2; 3].

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