On the BP Homology and Cohomology of $P^{2n} \wedge P^{2m}$

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1. Statement of Results

Let BP be the Brown–Peterson spectrum associated with the prime 2 and let $\mathrm{BP}_*(\ )$ and $\mathrm{BP}^*(\ )$ be the corresponding reduced homology and cohomology theories. Let P^{2n} be the $2n$-dimensional real projective space. There is a Künneth short exact sequence due to Landweber [3] for both $\mathrm{BP}_*(P^{2n} \wedge P^{2m})$ and $\mathrm{BP}^*(P^{2n} \wedge P^{2m})$ which is split exact in this case. For instance, for the BP-cohomology one has

$$\mathrm{BP}^*(P^{2n} \wedge P^{2m}) = \Sigma^{-1} \mathrm{Tor}_{\mathrm{BP}^*}(\mathrm{BP}^*(P^{2n}), \mathrm{BP}^*(P^{2m})) \oplus \mathrm{BP}^*(P^{2n}) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m}). \tag{1}$$

The tensor product module is well understood. It is the ideal generated by xy in the polynomial algebra $\mathrm{BP}^*[x,y]$ modulo the ideal $(([2]x)y, x([2]y))$, where $[2]x$ denotes the two-series in x. Furthermore, the tensor product has been computed as an abelian group in each degree larger than $2\max\{m,n\}$ [1; 2]. This computation has led to a strong non-immersion theorem for real projective spaces into Euclidean spaces [2].

Our goal in this note is to compute the Tor groups as BP-modules. We shall prove the following propositions.

PROPOSITION 1. $\mathrm{BP}^{\mathrm{odd}}(P^{2n} \wedge P^{2m}) = \Sigma^{-1} \mathrm{Tor}_{\mathrm{BP}^*}(\mathrm{BP}^*(P^{2n}), \mathrm{BP}^*(P^{2m}))$ *is isomorphic as a* BP^**-module to a copy of* $\Sigma^{2\max\{m,n\}-1}\mathrm{BP}^*(P^{2\min\{m,n\}})$.

PROPOSITION 2. $\mathrm{BP}_{\mathrm{odd}}(P^{2n} \wedge P^{2m}) = \Sigma^1 \mathrm{Tor}^{\mathrm{BP}_*}(\mathrm{BP}_*(P^{2n}), \mathrm{BP}_*(P^{2m}))$ *is isomorphic as a* BP_**-module to a copy of* $\Sigma^2 \mathrm{BP}_*(P^{2\min\{m,n\}})$.

We shall prove Proposition 1 in detail. The dual computation for homology follows the same line of proof and only a brief sketch will be given. As a by-product of the computation we get all of the v_1-torsion of the tensor product. Explicitly, we have the following corollary.

COROLLARY 9. *The* v_1*-torsion submodule of* $\mathrm{BP}^*(P^{2n}) \otimes_{\mathrm{BP}^*} \mathrm{BP}^*(P^{2m})$ *is the ideal generated by* $xy(x-y)$.

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