Commutativity Theorems for Banach Algebras

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1. Introduction

A number of theorems in ring theory, mostly due to Herstein, are devoted to showing that certain rings must be commutative as a consequence of conditions which are seemingly too weak to imply commutativity. For surveys of work in this area see [7, Chap. 3] and [10, Chap. X]. Our first aim is to show that in the special case of a Banach algebra some of these results may be sharpened.

Consider the following theorem of Herstein [4, p. 411]. A ring R is commutative if (a) there are no nonzero nil ideals and (b) for each x and y in R there is a positive integer $n(x,y)$ such that $x^{n(x,y)}$ permutes with y.

Let A be a Banach algebra which satisfies the following weakening of (b). Suppose (c) there exists a nonvoid open subset G of A, where for each x and y in G there are positive integers $m=m(x,y)$ and $n=n(x,y)$ such that $x^m y^n = y^n x^m$. If A has a two-sided approximate identity then A is commutative. In general, A need not be commutative but there must exist a positive integer r such that x^r lies in the center of A for all $x \in A$. If A has no nonzero nilpotent ideals, then A is commutative.

Consider also the theorem of Herstein [4, p. 412] which states that a ring R is commutative if for each x and y in R there is a positive integer $n(x,y)>1$ such that $x^{n(x,y)}-x$ permutes with y. For a Banach algebra A we show that $a \in A$ lies in the center if there is a nonvoid open set G where, for each $x \in G$, we have a positive integer $n(x)>1$ so that $x^{n(x)}-x$ permutes with a.

In Section 3 we present theorems in this spirit for Banach *-algebras. Let A be a Banach *-algebra with continuous involution and no nonzero nilpotent ideals (as when A is semi-simple). It is shown that either A is commutative or the set of $x \in A$, where x^n is normal for *no* positive integer n, is dense in A. If A is unital then the requirement on nilpotent ideals can be dropped. Other related results are obtained.

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