Associate Harmonic Immersions in 3-Space

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1. Introduction

The immersion of a surface S with definite prescribed metric g is *harmonic* into Euclidean 3-space $E^{3,0}$ (or into Minkowski 3-space $E^{3,1}$) if and only if the three coordinate functions of the immersion satisfy Laplace's equation with respect to coordinates isothermal for g. Similarly, the immersion of a surface S with indefinite prescribed metric g is harmonic into $E^{3,0}$ (or into $E^{3,1}$) if and only if the three coordinate functions of the immersion satisfy the wave equation with respect to coordinates isothermal for g.

Since the immersion of a surface S with definite or indefinite prescribed metric g is harmonic into $E^{3,0}$ if and only if it is harmonic into $E^{3,1}$, we refer throughout this paper to a harmonic immersion $\mathcal{Z}: (S,g) \to E^{3,j}$ where the index j can assume either value 0 or 1. The results below are meant, in part, to illustrate that the Minkowski geometry of a harmonic immersion $\mathcal{Z}: (S,g) \to E^{3,j}$ is at least as interesting as its Euclidean geometry.

An immersion $\mathcal{Z}: S \to E^{3,0}$ is *minimal* if and only if $\mathcal{Z}: (S, I^0) \to E^{3,j}$ is harmonic, where I^0 is the metric induced on S by $E^{3,0}$. Similarly, an immersion $\mathcal{Z}: S \to E^{3,1}$ is minimal if and only if $\mathcal{Z}: (S, I^1) \to E^{3,j}$ is harmonic, where I^1 is the metric induced on S by $E^{3,1}$. Since I^0 and I^1 are seldom proportional, the harmonic immersions into $E^{3,j}$ which are minimal into $E^{3,0}$ differ from those minimal into $E^{3,1}$.

In this paper, we are most concerned with harmonic immersions $\mathcal{Z}: (S,g) \to E^{3,j}$ with indefinite prescribed metric g. We look at harmonic immersions $\mathcal{Z}: (S,g) \to E^{3,j}$ with definite prescribed metric g solely to compare and contrast results. When g is definite, the properties of harmonic $\mathcal{Z}: (S,g) \to E^{3,j}$ tend to generalize the behavior of minimal $\mathcal{Z}: S \to E^{3,0}$ (see [3]). When g is indefinite, the properties of harmonic $\mathcal{Z}: (S,g) \to E^{3,j}$ tend to generalize the behavior of timelike minimal $\mathcal{Z}: (S,g) \to E^{3,1}$ (see [5] and [6]).

In Section 3 we define associate families of harmonic immersions into $E^{3,j}$. For definite g, the construction imitates the classical definition of associate families of minimal immersions in $E^{3,0}$ (see [2]). For indefinite g, the construction specializes to the definition of associate families of timelike minimal immersions into $E^{3,1}$, which are studied more closely in [6].

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