Totally Umbilic Riemannian Foliations

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1. Introduction

On a Riemannian manifold, a foliation with leaves of dimension $p \geq 2$ is said to be *totally umbilic* if its leaves are totally umbilic submanifolds. An obvious example is that of Euclidean space $\mathbf{E}^n$, minus one point, foliated by concentric spheres. Further examples are provided by *totally geodesic* foliations—that is, foliations whose leaves are totally geodesic submanifolds. Whereas totally geodesic foliations have received a good deal of attention in the literature (see the references in [6]), there are surprisingly few works on totally umbilic foliations (see nevertheless [18], [3], [4], [11]).

Considering the foliation F of $\mathbf{E}^n \setminus \{\bigstar\}$ by concentric spheres, one will notice that it can be made totally geodesic by suitably changing the metric. Indeed, $\mathbf{E}^n \setminus \{\bigstar\}$ is diffeomorphic to $S^{n-1} \times \mathbf{R}$ via the obvious map that sends the leaves of F to the submanifolds $S^{n-1} \times \{pt\}$. A foliation F is said to be *umbilicalisable* [7] (resp. *geodesible*) if there exists a Riemannian metric on the ambient manifold for which F is totally umbilic (resp. totally geodesic). It is not hard to construct umbilicalisable foliations that are not geodesible. One obstruction is that a totally geodesic foliation is necessarily *harmonic;* that is, its leaves are minimal submanifolds. A foliation F is said to be *taut* if there exists a Riemannian metric on the ambient manifold for which F is harmonic. Clearly, a foliation is totally geodesic if and only if it is totally umbilic and harmonic for the same metric. It is less obvious that a taut umbilicalisable foliation is necessarily geodesible. In his thesis [7], Carrière made the following conjecture.

CONJECTURE 1 (Carrière). *Every codimension-*1 *taut umbilicalisable foliation on a compact manifold is geodesible.*

Carrière proved this conjecture for codimension-1 Riemannian foliations. In this paper we consider the case of Riemannian foliations of arbitrary codimension. Recall that a foliation F on a manifold M is *Riemannian* (see [20], [17], [13]) if there exists a Riemannian metric g on the transverse bundle TM/TF for which the leaves are locally equidistant. This amounts to saying that for a codimension-q foliation F defined by local submersions

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