The Action of S_n on the Components of the Hodge Decomposition of Hochschild Homology

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1. Background

Let k be a field of characteristic 0, let A be an associative k-algebra, and let M be an A-bimodule. Define $C_n(A;M)$ to be $M\otimes A^{\otimes n}$ (all tensor products over k) and define $b_n: C_n(A;M)\to C_{n-1}(A;M)$ by

$$b_n(m\otimes a_1\otimes\cdots\otimes a_n)=ma_1\otimes a_2\otimes\cdots\otimes a_n+(-1)^n a_n m\otimes a_1\otimes\cdots\otimes a_{n-1}$$
$$+\sum_{i=1}^{n-1}(-1)^i m\otimes a_1\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n.$$

It is easy to check that $b_n\circ b_{n+1}=0$, so that $\operatorname{im} b_{n+1}$ is contained in $\ker b_n$. The *Hochschild homology of A with coefficients in M* is defined by

$$H_n(A;M)=\frac{\ker b_n}{\operatorname{im} b_{n+1}}.$$

The symmetric group S_n acts on $C_n(A;M)$ by

$$\sigma\cdot(m\otimes a_1\otimes\cdots\otimes a_n)=m\otimes a_{\sigma^{-1}1}\otimes\cdots\otimes a_{\sigma^{-1}n}.$$

Define a *splitting sequence* $(f_n)_{n=1}^{\infty}$ to be a sequence of elements $f_n\in k[S_n]$ such that

$$b_n f_n\alpha=f_{n-1}b_n\alpha \tag{1.1}$$

for all $\alpha\in C_n(A;M)$, all associative k-algebras A, and all A-bimodules M. Given a splitting sequence (f_n) one can define $I_n(A;M)$ and $K_n(A;M)$ to be the image and kernel of $C_n(A;M)$ under f_n. Then both $(I_*(A;M),b_*)$ and $(K_*(A;M),b_*)$ are subcomplexes of $(C_*(A;M),b_*)$ which then yield new homology theories.

One can obtain a trivial splitting sequence by letting f_n be the identity in S_n. In general this is the only splitting sequence. However, under the assumption that A is commutative and that M is a symmetric bimodule ($a\cdot m=m\cdot a$ for all $a\in A$ and $m\in M$), there do exist nontrivial splitting sequences.

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