Invariant Subspaces in Banach Spaces of Analytic Functions

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Introduction

Let $\mathbf{D}$ denote the open unit disc $\{z \in \mathbf{C}: |z| < 1\}$, let $\mathbf{T}$ denote the boundary $\{z \in \mathbf{C}: |z| = 1\}$, and let $\mathbf{D}^-$ denote the closure of $\mathbf{D}$. By $A = A(\mathbf{D})$ we denote the disc algebra, consisting of those continuous functions on $\mathbf{D}^-$ that are analytic in $\mathbf{D}$. We introduce also the Fréchet space $\mathcal{O}(\mathbf{D})$ of all holomorphic functions on $\mathbf{D}$, with the topology of uniform convergence on compact subsets, and the space $\mathcal{O}(\mathbf{D}^-)$ of (germs of) functions holomorphic on neighborhoods of $\mathbf{D}^-$, with the inductive limit topology. Strictly speaking the elements of $\mathcal{O}(\mathbf{D}^-)$ are equivalence classes, but we shall often regard them as individual functions. A *sequence* $\{f_k\}$ in $\mathcal{O}(\mathbf{D}^-)$ converges to a function f if and only if all the functions are analytic in some fixed open set U containing $\mathbf{D}^-$, with $f_k \to f$ uniformly on compact subsets of U.

Let X be a Banach subspace of $\mathcal{O}(\mathbf{D})$; more precisely, this means that X is a vector subspace of $\mathcal{O}(\mathbf{D})$ and X has a norm with respect to which it is a Banach space. We assume further that the injection map $X \to \mathcal{O}(\mathbf{D})$ is continuous, and that X contains $\mathcal{O}(\mathbf{D}^-)$ as a dense subspace. (The injection map, $\mathcal{O}(\mathbf{D}^-) \to X$, is automatically sequentially continuous; see the remarks following Lemma 1 below.) Since the point evaluation functionals at the points of $\mathbf{D}$ are continuous functionals on $\mathcal{O}(\mathbf{D})$ it follows that they are also continuous on X. (In fact, the continuity of these functionals is equivalent to the continuity of the injection map of X into $\mathcal{O}(\mathbf{D})$; see Proposition 1 of [6].)

Note that X must be separable. Indeed, every function in $\mathcal{O}(\mathbf{D}^-)$ is the limit, in the topology of $\mathcal{O}(\mathbf{D}^-)$, of a sequence of polynomials with rational coefficients.

Let $M(X)$ denote the space of multipliers of X, that is, the set of all those functions $\varphi \in \mathcal{O}(\mathbf{D})$ such that $\varphi X \subset X$. Using the closed graph theorem, one shows that if $\varphi \in M(X)$ then multiplication by φ is a bounded linear transformation on X. Also, by [10, Prop. 11] one has $M(X) \subset H^\infty$, and the supremum norm is less than or equal to the operator norm. Here H^∞ denotes the space of bounded analytic functions in $\mathbf{D}$ with the supremum norm. Using

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