Analytic Functions That Have Convex Successive Derivatives

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1. Introduction

In a series of papers [4]–[8], Shah and Trimble studied the family of univalent functions such that an infinite sequence of the successive derivatives (possibly each successive derivative) is univalent. Many of their results involved functions that map the unit disk onto domains that have certain geometric properties such as close-to-convexity or convexity. For example, Theorems 1 and 2 in [8] give necessary and sufficient conditions on β and $\{z_k\}_{k=1}^N$ (with each z_k on a given ray) so that the function f given by

$$f(z)=ce^{\beta z}\prod_{k=1}^{N}\left(1-\frac{z}{z_k}\right)$$

is close-to-convex (or convex) and has each successive derivative close-to-convex (or convex). They conjecture that the function e^z-1 has many extremal properties within the family of functions f that are analytic in $D\equiv\{|z|<1\}$, are normalized by $f(0)=0$ and $f'(0)=1$, and have the property that $f(D), f'(D), f''(D), \ldots$ are all convex. For example, they conjecture that for such an f with $f(z)=z+a_2z^2+\cdots$,

$$1-e^{-|z|}\le|f(z)|\le e^{|z|}-1 \quad\text{and}\quad |a_k|\le\frac{1}{k!}.$$

In [9] we showed that if f as above satisfies the conditions described and if, in addition, each coefficient a_k is positive, then $a_k\le 1/k!$ and then clearly $|f(z)|\le e^{|z|}-1$.

In [1], Barnard and Suffridge showed that if f is analytic in D, $f(0)=0$, $f'(0)=1$, and $f(D)$ and $f'(D)$ are convex, then the coefficients a_k, $k\ge2$, satisfy $|a_k|\le 4/(3k)$ with equality if and only if $f(z)=-\frac{z}{3}-\frac{4}{3}\log(1-z)$ (so that $f'(z)=1+\frac{4}{3}(z/(1-z))$ or a rotation $e^{-i\alpha}f(ze^{i\alpha})$ of this function). In this paper, we prove some general theorems concerning convexity of f when $f'(z)=1+2ag(z)$ and g is convex. We then study some extremal problems in K_n, where $K=K_0=\{f\colon f$ is analytic in D, $f(0)=0$, $f'(0)=1$, and $f(D)$ is convex$\}$, $K_{n+1}=\{f\in K_n\colon f^{(n+1)}(D)$ is convex or $f^{(n+1)}$ is constant$\}$, and $K_\infty=\bigcap_{n=0}^{\infty}K_n$. We find the sharp coefficient bounds in the family K_2, and

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