A Removable Set for Lipschitz Harmonic Functions

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1. Introduction

Let K be a compact set of d-dimensional space $\mathbf{R}^d$ $(d \geq 2)$. We denote the class of bounded harmonic functions defined on $\mathbf{R}^d \backslash K$ by $\mathcal{H}^\infty(K)$ and denote the class of those functions in $\mathcal{H}^\infty(K)$ that satisfy a Lipschitz condition of order α $(0 < \alpha \leq 1)$ by $\mathcal{H}_\alpha^\infty(K)$. The following result on the removable singularities of bounded harmonic functions is well known (see [1, Chap. VII] and [6, Chap. III]): Every $f \in \mathcal{H}^\infty(K)$ is extendable harmonically across K if and only if K has a zero capacity.

Regarding the class of $\mathcal{H}_\alpha^\infty(K)$ for $0 < \alpha < 1$, Carleson [1] proved that K is removable if and only if $\Lambda_{d-2+\alpha}(K) = 0$, where $\Lambda_{d-2+\alpha}$ denotes the $(d-2+d)$-dimensional Hausdorff measure.

The motivation for this paper arises primarily from [7], where the author studied the removable singularities of Lipschitz analytic functions. Our purpose here is to show that there can be a set K with $\Lambda_{d-1}(K) > 0$ even though K is removable for the class $\mathcal{H}_1^\infty(K)$.

This contrasts with the following analogous result obtained for analytic functions: For all α $(0 < \alpha \leq 1)$, a compact set $K \subseteq \mathbf{C}$ is removable for the class of bounded analytic functions satisfying a Lipschitz condition of order α if and only if $\Lambda_{1+\alpha}(K) = 0$ (see [2], [7]). Arguments used in this paper are largely based on a related paper by Garnett [4] on removable singularities of bounded analytic functions.

2. A Multi-Dimensional Cantor Set

In this section we define a d-dimensional Cantor set K with $\Lambda_{d-1}(K) > 0$. First, we form a linear Cantor set E with ratio $\lambda = 2^{-d/(d-1)}$, using an inductive method as in the construction of the well-known one-third Cantor set. We obtain $E = \bigcap_{n=0}^\infty E_n$, where $E_0 = [0,1]$ and E_n $(n = 0, 1, 2, \ldots)$ contains 2^n disjoint intervals of length equal to $2^{-nd/(d-1)}$. Define

$$K_n = \prod_{i=1}^d E_n, \quad n = 0, 1, 2, \ldots, \quad \text{and} \quad K = \bigcap_{n=0}^\infty K_n.$$

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