Bounds for the Degrees in the Division Problem

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0. Introduction

One of the basic questions in computational algebra is to find sharp bounds for the complexity of the possible algorithms solving the following problem. Let $f_1, \dots, f_m \in \mathbf{C}[z]$ $(z = (z_1, \dots, z_n))$, and let I be the ideal they generate. Assuming that $f \in I$, what can be said about the polynomials $q_1, \dots, q_m$ that satisfy the equation

$$(*) \qquad f = q_1 f_1 + \cdots + q_m f_m?$$

To be more concrete, let us assume that $\max\{\deg f, \deg f_j, (1 \le j \le m)\} = D$. Can we find q_j whose degrees are relatively small—for example, bounded by D^n?

In the case where the homogenized polynomials ${}^h f_1, \dots, {}^h f_m$ of $f_1, \dots, f_m$ define a complete intersection variety in $\mathbf{C}^{n+1}$, such bound is correct. In fact, one can find polynomials $q_1, \dots, q_m$ such that

$$\max_{1 \le j \le m} \deg(f_j q_j) \le \deg f + d_1 \dots d_m,$$

where $d_j = \deg f_j$. This is a consequence of a result of Macaulay dating from the beginning of the century. It follows from the fact that for a locally regular sequence the exponent in the local Nullstellensatz is bounded by the product of the degrees (see [14]).

It is surprising to find that such an estimate is false in general. An example of Mayr–Meyer [9] shows that for any $D \ge 5$, $k \ge 1$ and $n = 10k$, there are $n+1$ polynomials $f_1, \dots, f_{n+1} \in \mathbf{C}[z]$ such that $z_1 \in I$, and that if $q_1, \dots, q_{n+1} \in \mathbf{C}[z]$ satisfy $z_1 = q_1 f_1 + \cdots + q_{n+1} f_{n+1}$ then $\max \deg q_j > (D-2)^{2^{k-1}}$. This implies that, in general, the complexity of any algorithm capable of solving this kind of problem must be doubly exponential. This applies in particular to the algorithms that construct the standard (or Groebner) bases. The reason is that, as soon as such a basis is known, the problem of deciding whether $f \in I$ and finding the q_j can be solved immediately.

There are several ways to obtain division formulas with better bounds for the degrees of the q_j. For instance, in comparing $(*)$ with Hilbert's Null-

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