On the Fourier Series of a Step Function

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Introduction

In recent years there has been a growth of interest in the interplay between Fourier series and harmonic mappings of the disc, with particular emphasis on connections with the topology of curves. Examples of this are:

(1) Hall's work [3] establishing the correct value of the Heinz constant concerning the Fourier coefficients of a circle mapping;
(2) the paper by Clunie and the author [2] taking the first steps towards establishing a general theory of univalent harmonic functions;
(3) several papers by Hengartner and Schober, including one containing a harmonic version of the Riemann mapping theorem [4] and a second studying a class of open harmonic mappings and their boundary behaviour [5]; and
(4) the author's paper [7] proving Shapiro's conjecture on the Fourier coefficients of an N-fold mapping of the circle and the discovery of an unexpected connection with certain classically defined multivalent analytic functions.

A result which continues to dominate this theory, and which has received a number of new proofs, is that due to Kneser [6], Rado, and (independently) Choquet [1] concerning the univalence of the harmonic extension into the disc of a homeomorphism of the circle onto a convex curve. Indeed, much of the current work either uses this result in a direct way or develops a variety of generalizations and analogous ideas. This function-theoretic development of harmonic mappings in the plane leads naturally to a variety of extremal problems. In a number of cases the extremal functions turn out to have boundary values which are step functions on the unit circle. For example, the extremal function giving the sharp value for the Heinz constant is

$$f(t)=\omega_k \quad (2\pi k/3<t<2\pi(k+1)/3,\ k=0,1,2),$$

where $\omega_k=\exp(2\pi ik/3)$ [3]. The harmonic extension of this function is a homeomorphism of the disc onto the interior of the triangle formed by the vertices ω_k. Furthermore, Hengartner and Schober [5] show that step-function solutions arise naturally as a class of solutions of their differential equation

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